

# EXPLICIT INDUCTION PRINCIPLE AND SYMPLECTIC-ORTHOGONAL THETA LIFTING

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**ABSTRACT.** First, an explicit version of induction principle is formulated to compute the local theta correspondence for  $(O(p, q), Sp(2n, \mathbb{R}))$  with  $p + q$  even: when  $p + q \leq 2n$ , the Langlands parameters of the theta  $(n+k)$ -lift of a representation of  $O(p, q)$  is read off from the parameters of its theta  $n$ -lift, if the  $n$ -lift is nonzero; similarly when  $p + q \geq 2n + 2$ , a nonzero theta  $(p, q)$ -lift of a representation of  $Sp(2n, \mathbb{R})$  determines its theta  $(p+k, q+k)$ -lift explicitly. Secondly, after reducing computations by our explicit induction principle, a complete and explicit description of the local theta correspondence is obtained for all the dual pairs  $(O(p, q), Sp(2n, \mathbb{R}))$  with  $p + q = 4$ . Our strategy is to determine the theta lifts under consideration by their infinitesimal characters and lowest  $K$ -types.

## 1. INTRODUCTION

**1.1. Background.** To understand the unitary duals of reductive Lie groups, a large number of unitary representations were constructed via the local theta correspondence for reductive dual pairs (cf. [How87, Li89]). For example, for  $(G, G') = (O(p, q), Sp(2n, \mathbb{R}))$  which is a reductive dual pair in  $Sp(2n(p+q), \mathbb{R})$ , let  $p + q$  be even (so that the metaplectic  $\mathbb{C}^\times$ -cover splits on  $G \cdot G'$ ) and  $p + q \leq n$  (so that this pair is in the stable range with  $G$  the smaller member), then the local theta correspondence gives an injection from the unitary dual of  $G$  to that of  $G'$ . Varying  $(p, q)$ , we get families of singular unitary representations of  $G'$  which comes from the unitary duals of various  $G$  of smaller size.

To give the Langlands parameters of the unitary representations thus obtained, and for other applications to automorphic forms, it is of great interest to determine the theta correspondence explicitly in terms of Langlands parameters. On this subject there have been a number of papers. For reductive dual pairs of type II over archimedean fields or complex groups in dual pairs, the local theta correspondence is completely and explicitly described in [Mœg89, AB95, LPTZ03]. For other reductive dual pairs of type I over archimedean fields, explicit results up to now are mainly for compact cases (see [EHW83]) and “(almost) equal rank” cases (see [AB98, Pau98, Pau05]), with the exception of [LTZ01] and [LPTZ03] for some non-equal-rank cases. For general cases of those dual pairs, the local theta correspondence seems very far from a complete and explicit description.

In the present paper, we give some explicit results about the local theta correspondence for  $(O(p, q), Sp(2n, \mathbb{R}))$  with  $p + q$  even. More precisely, we formulate an explicit version of induction principle, and then use it to compute the full correspondence for all  $n$  when  $p + q = 4$ . Similar arguments should prove similar explicit induction principle for other reductive dual pairs of type I over archimedean fields.

**1.2. Description of the problem.** In this paper we investigate the local theta correspondence for reductive dual pairs  $(G, G') = (O(p, q), Sp(2n, \mathbb{R}))$  with  $p + q$  even.

First recall how to interpret the local theta correspondence for  $(G, G')$  as a correspondence between representations of  $G$  and representations of  $G'$ . The metaplectic  $\mathbb{C}^\times$ -cover

$$1 \rightarrow \mathbb{C}^\times \rightarrow Mp(2n(p+q), \mathbb{R}) \rightarrow Sp(2n(p+q), \mathbb{R}) \rightarrow 1$$

splits on  $G \cdot G'$  as  $p+q$  is even, with a particular splitting map as described in [Kud94]. This splitting map indeed fixes an embedding  $G \cdot G' \hookrightarrow Mp(2n(p+q), \mathbb{R})$ . Via this embedding, the *Segal-Shale-Weil oscillator representation*  $\omega$  (associated to the character of  $\mathbb{R}$  given by  $t \mapsto e^{2\pi i t}$ ) is restricted to  $G \cdot G'$ . Let  $\mathcal{R}(G)$  denote the *admissible dual* of  $G$ , i.e., the set of infinitesimal equivalence classes of continuous irreducible admissible representations of  $G$ . By abuse of notation, for a class  $\pi \in \mathcal{R}(G)$ ,  $\pi$  also denotes a representation in this class. Let  $\mathcal{R}(G, \omega)$  be the set of elements  $\pi$  of  $\mathcal{R}(G)$  such that there exists a nontrivial  $G$ -intertwining continuous linear map  $\omega \rightarrow \pi$ . Similar notations hold for  $G'$  and  $G \cdot G'$ . [How89] gives a one-to-one correspondence  $\mathcal{R}(G, \omega) \leftrightarrow \mathcal{R}(G', \omega)$ , which is usually called the *local theta correspondence* or *Howe duality correspondence*.

For  $\pi \in \mathcal{R}(G)$  and  $\pi' \in \mathcal{R}(G')$ , they correspond if and only if  $\pi \otimes \pi' \in \mathcal{R}(G \cdot G', \omega)$ . In this case we write  $\theta_n(\pi) = \pi'$  and  $\theta_{p,q}(\pi') = \pi$ . If  $\pi \notin \mathcal{R}(G, \omega)$ , write  $\theta_n(\pi) = 0$ . Similarly write  $\theta_{p,q}(\pi') = 0$  if  $\pi' \notin \mathcal{R}(G', \omega)$ . Then  $\theta_n(\pi)$  is called the *theta  $n$ -lift* of  $\pi$ , and the map  $\theta_n : \mathcal{R}(G) \rightarrow \mathcal{R}(G', \omega) \cup \{0\}$  is called the *theta  $n$ -lifting* for  $G$ . Similarly  $\theta_{p,q}(\pi')$  is called the *theta  $(p, q)$ -lift* of  $\pi'$ , and  $\theta_{p,q} : \mathcal{R}(G') \rightarrow \mathcal{R}(G, \omega) \cup \{0\}$  is called the *theta  $(p, q)$ -lifting* for  $G'$ .

Our goal is to determine  $\theta_n : \mathcal{R}(G) \rightarrow \mathcal{R}(G', \omega) \cup \{0\}$  explicitly. Here “explicitly” means to parametrize  $\mathcal{R}(G)$  and  $\mathcal{R}(G')$  in terms of *Langlands parameters*. We choose Vogan’s version of Langlands classification in [Vog84], which is explicitly described in [Pau05] and recollected in following Subsection 2.6 and 2.7. Roughly speaking,  $\pi \in \mathcal{R}(G)$  is parametrized in the form  $\pi = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa)$ , and  $\pi' \in \mathcal{R}(G')$  in the form  $\pi' = \pi(\lambda_d, \Psi, \mu, \nu, \varepsilon, \kappa)$ .

For  $(G, G') = (O(p, q), Sp(2n, \mathbb{R}))$  with  $p+q$  even, [Mœg89, Pau05] determines the local theta correspondence explicitly when  $p+q = 2n$  or  $2n+2$  (equal rank or almost equal rank cases). The present paper will give two new results for this family of dual pairs:

- (i) an explicit version of induction principle for the local theta correspondence;
- (ii) a full and explicit description of the local theta correspondence for all  $n, p, q$  with  $p+q = 4$ .

Our approach is very elementary by analysis on the information of infinitesimal character and  $K$ -types under the correspondence. The main results and basic ideas are outlined as follows.

**1.3. Explicit induction principle.** For  $\pi \in \mathcal{R}(G)$  and  $\pi' \in \mathcal{R}(G')$ , Kudla’s *persistence principle* states that ([Mœg89] I.9): if  $\theta_n(\pi) \neq 0$  then  $\theta_{n+1}(\pi) \neq 0$ ; if  $\theta_{p,q}(\pi') \neq 0$  then  $\theta_{p+1,q+1}(\pi') \neq 0$ . To determine  $\theta_{n+1}(\pi)$  from a given nonzero  $\theta_n(\pi)$  (respectively,  $\theta_{p+1,q+1}(\pi')$  from a given nonzero  $\theta_{p,q}(\pi')$ ), one of the most powerful tools available is the “induction principle” developed by [Kud86, Mœg89, AB95, Pau05]. In this paper, it is extended to an explicit version for  $(O(p, q), Sp(2n, \mathbb{R}))$  with  $p+q$  even, in terms of Langlands parameters described in Subsection 2.6 and 2.7.

**Theorem 1** (Explicit induction principle). *Suppose  $p+q$  is even. Let  $1 \leq k \in \mathbb{Z}$ . For  $x = (x_1, \dots, x_m) \in \mathbb{C}^m$  and  $y = (y_1, \dots, y_l) \in \mathbb{C}^l$ , write  $(x \mid y) = (x_1, \dots, x_m, y_1, \dots, y_l) \in \mathbb{C}^{m+l}$ .*

- (1) *If  $\pi \in \mathcal{R}(O(p, q))$  and  $0 \neq \theta_n(\pi) = \pi(\lambda_d, \Psi, \mu, \nu, \varepsilon, \kappa)$  with  $2n \geq p+q$ , then*

$$\begin{aligned} \theta_{n+k}(\pi) &= \pi(\lambda_d, \Psi, \mu, \nu, (\varepsilon \mid ((-1)^{\frac{p-q}{2}}, (-1)^{\frac{p-q}{2}}, \dots, (-1)^{\frac{p-q}{2}})), \\ &\quad (\kappa \mid (1+n-\frac{p+q}{2}, 2+n-\frac{p+q}{2}, \dots, k+n-\frac{p+q}{2}))) \end{aligned}$$

*with a possible modification.*

(2) If  $\pi' \in \mathcal{R}(Sp(2n, \mathbb{R}))$  and  $0 \neq \theta_{p,q}(\pi') = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa)$  with  $p + q \geq 2n + 2$ , then  $\zeta = \xi = 1$  and

$$\begin{aligned} \theta_{p+k, q+k}(\pi') &= \pi_1(\lambda_d, 1, \Psi, \mu, \nu, (\varepsilon \mid (1, 1, \dots, 1))), \\ &(\kappa \mid (\frac{p+q}{2} - n, 1 + \frac{p+q}{2} - n, \dots, k - 1 + \frac{p+q}{2} - n))) \end{aligned}$$

with a possible modification.

The above “possible modification” is: if the resulting parameters contain some entries  $\varepsilon_i \neq \varepsilon_j$  and  $\kappa_i = \pm \kappa_j$ , delete these  $\varepsilon_i, \varepsilon_j, \kappa_i, \kappa_j$  from  $(\varepsilon, \kappa)$ , and add entries  $(0, 2\kappa_i)$  into  $(\mu, \nu)$ .

This theorem will be proved as special cases of Theorem 19 and Theorem 25. Indeed, [Pau05] already pointed out the result (1) when  $p + q = 2n$  and the result (2) when  $p + q = 2n + 2$ . Our observation is that Paul’s idea can be generalized to more cases, due to certain patterns of the Langlands parameters of theta lifts in these cases (see Lemma 17 and 24). This is the first main result of the present paper, and we call it the “explicit induction principle”.

By the nonvanishing of theta liftings in the stable range [PP08],  $\theta_{p+q}(\pi) \neq 0$  for all  $\pi \in \mathcal{R}(G)$ . If  $\theta_{p+q}(\pi)$  is given explicitly, then  $\theta_{p+q+k}(\pi)$  is read off by the explicit induction principle. For fixed  $(p, q)$  with  $p + q$  even, the explicit induction principle is strong enough to reduce the computation of  $\theta_n$  for all  $n$  to only finitely many  $n \leq p + q$ .

**1.4. Explicit theta correspondence when  $p + q = 4$ .** We will determine  $\theta_n : \mathcal{R}(O(p, q)) \rightarrow \mathcal{R}(Sp(2n, \mathbb{R})) \cup \{0\}$  explicitly in terms of Langlands parameters for all  $n$  when  $p + q = 4$ . Notice that by our explicit induction principle, it suffices to compute  $\theta_3$  and  $\theta_4$ , as  $\theta_1$  and  $\theta_2$  are explicitly given in [Pau05] (and written explicitly in Appendix B). Moreover, for  $\pi \in \mathcal{R}(O(p, q))$ , we see that  $\theta_{p+q-1}(\pi) \neq 0$  if  $\pi$  is not the *determinant character*  $\det$ . Then the explicit induction principle further reduces the computation to theta 4-lifts of  $\det$  and theta 3-lifts of other  $\pi \in \mathcal{R}(G)$  such that  $\theta_2(\pi) = 0$ . For every such  $\pi$  (or  $\det$ ), we shall list the Langlands parameters of it, and calculate two invariants of it: the *infinitesimal character* and the set  $\mathcal{A}(\pi)$  (or  $\mathcal{A}(\det)$ ) of all its *lowest  $K$ -types* ( $K$ -types with minimal norm in the sense of [Vog79]), by algorithms in Subsection 2.8 and Appendix A. For its theta lift, we shall also try to calculate these two invariants. After that we will see very fortunately that these two invariants are good enough to determine the theta lift for almost all the cases under consideration with one exception.

The correspondence of infinitesimal characters under theta correspondence is clearly known by [Prz96] (see Subsection 2.8). So the infinitesimal characters of  $\theta_3(\pi)$  and  $\theta_4(\det)$  are got.

We can obtain  $\mathcal{A}(\theta_3(\pi))$  in three steps:  $\mathcal{A}(\pi) \rightsquigarrow \mathcal{D}(\pi) \rightsquigarrow \mathcal{D}(\theta_3(\pi)) \rightsquigarrow \mathcal{A}(\theta_3(\pi))$ .

(1) We have  $\mathcal{A}(\theta_3(\pi)) \subseteq \mathcal{D}(\theta_3(\pi))$ , where  $\mathcal{D}(\theta_3(\pi))$  is the set of all  *$K$ -types of minimal degree* in  $\theta_3(\pi)$  in the sense of [How89]. (This fact is shown in [Pau05] for  $p + q \leq 2n$ , and is contained in the following Lemma 17). So  $\mathcal{A}(\theta_3(\pi)) = \{\sigma \in \mathcal{D}(\theta_3(\pi)) \text{ with minimal norm}\}$ .

(2) We know  $\mathcal{D}(\pi) \leftrightarrow \mathcal{D}(\theta_3(\pi))$  under the correspondence of  $K$ -types in the space of joint harmonics, which is explicitly expressed in Proposition 4.

(3) By definition  $\mathcal{D}(\pi) = \{K\text{-types of minimal degree in } \pi\}$ . We only need to check the occurrence in  $\pi$  of the  $K$ -types with degrees  $\leq$  the minimal degree of  $K$ -types in  $\mathcal{A}(\pi)$ . This can be done by Frobenius reciprocity and explicit models of  $\pi$ .

Similarly, we get  $\mathcal{A}(\theta_4(\det))$ , which is easier since  $\det$  contains only one  $K$ -type.

**Theorem 2.** *Let  $\pi \in \mathcal{R}(O(p, q))$  with  $p+q = 4$ . Let  $\det$  be the determinant character of  $O(p, q)$ .*

(1) *The infinitesimal character of  $\theta_4(\det)$  and the set  $\mathcal{A}(\theta_4(\det))$  determine a unique element in  $\mathcal{R}(Sp(8, \mathbb{R}))$ , with Langlands parameters written explicitly in Section 5.*

(2) *If  $\theta_2(\pi) = 0$ ,  $\pi \neq \det$  or  $\pi_{-1}(0, 1, \emptyset, 0, 0, (1, 1), (0, 2))$ , then the infinitesimal character of  $\theta_3(\pi)$  and the set  $\mathcal{A}(\theta_3(\pi))$  determine a unique element in  $\mathcal{R}(Sp(6, \mathbb{R}))$ , with Langlands parameters written explicitly in Section 6.*

*Remark.* For the exceptional case  $\pi = \pi_{-1}(0, 1, \emptyset, 0, 0, (1, 1), (0, 2)) \in \mathcal{R}(O(2, 2))$ , there are exactly two elements in  $\mathcal{R}(Sp(6, \mathbb{R}))$  with the desired infinitesimal character and set of lowest  $K$ -types. It is not hard to figure out which of them lies in the local theta correspondence.

Theorem 2 is checked by tedious but elementary case-by-case calculation in Section 5, 6 and Appendix C. Appendix C indeed lists all  $\pi' \in \mathcal{R}(Sp(6, \mathbb{R}))$  with infinitesimal character  $(\beta, 0, 1)$  for all  $\beta \in \mathbb{C}$ , and gives  $\mathcal{A}(\pi')$  explicitly case by case. In this way, together with our explicit induction principle, we will determine the local theta correspondence for  $(O(p, q), Sp(2n, \mathbb{R}))$  for all  $n$  when  $p+q = 4$ , explicitly in terms of Langlands parameters.

## 2. PARAMETRIZATION

In this section we introduce some auxiliary notations and the parametrizations for  $K$ -types, representations, and infinitesimal characters. The advantage of our parametrizations is that explicit algorithms are given to calculate lowest  $K$ -types and infinitesimal characters from Langlands parameters.

**2.1. Notations for  $K$ -types.** For a compact Lie group  $K$ , a  $K$ -type means an irreducible representation (up to equivalence) of  $K$ . Let  $\widehat{K}$  denote the set of all  $K$ -types. A  $K$ -type is automatically finite-dimensional and unitary, so  $\widehat{K} = \mathcal{R}(K)$ .

Let  $H$  be a reductive Lie group with a maximal compact subgroup  $K_H$ . We refer to  $K_H$ -types as  $K$ -types for  $H$ , or simply as  $K$ -types if the group  $H$  is clearly understood. Let  $\pi$  be an admissible representation of  $H$ . Recall that “ $\pi$  is admissible” means the *multiplicity*  $m(\sigma, \pi) = \dim \text{Hom}_{K_H}(\sigma, \pi)$  of  $\sigma$  in  $\pi$  is finite for any  $K_H$ -type  $\sigma$ . When  $m(\sigma, \pi) > 0$ , we say that  $\sigma$  occurs in  $\pi$ , or  $\sigma$  is a  $K$ -type of  $\pi$ , or  $\pi$  contains  $\sigma$ .

For  $(G, G') = (O(p, q), Sp(2n, \mathbb{R}))$ , take the standard maximal compact subgroups  $O(p) \times O(q)$  and  $U(n)$  of  $O(p, q)$  and  $Sp(2n, \mathbb{R})$  respectively. Henceforth, for a Lie group, the corresponding lower case Gothic letter is used to indicate the complexified Lie algebra of it.

**2.2. Parametrization for  $U(n)$ -types.** For  $K = U(n)$ , take the maximal torus  $T = U(1)^n = \text{diag}(U(1), \dots, U(1))$ , and the standard system of positive roots

$$\Delta^+(\mathfrak{t}, \mathfrak{k}) = \{e_i - e_j \mid 1 \leq i < j \leq n\}.$$

Write each weight in  $\mathfrak{t}_0^*$  as the  $n$ -tuple of coefficients under the basis  $e_1, \dots, e_n$ . A  $U(n)$ -type is parametrized by its *highest weight*  $(a_1, a_2, \dots, a_n)$  with integers  $a_1 \geq a_2 \geq \dots \geq a_n$ .

**2.3. Parametrization for  $O(p) \times O(q)$ -types.** Notice that

$$\widehat{O(p)} \times \widehat{O(q)} = \{\sigma \otimes \tau \mid \sigma \in O(p), \tau \in O(q)\} \simeq \widehat{O(p)} \times \widehat{O(q)}.$$

**Lemma 3** ([Wey39]). *Embed  $O(p)$  in  $U(p)$  as  $O(p) = U(p) \cap GL(p, \mathbb{R})$ . Given an arbitrary  $\sigma \in \widehat{O(p)}$ , there exists a unique  $\lambda \in \widehat{U(p)}$  such that the  $O(p)$ -module generated by the highest weight vectors of  $\lambda$  is equivalent to  $\sigma$ . Moreover, these  $\lambda$  obtained from  $\widehat{O(p)}$  are exactly those parametrized as  $(b_1, b_2, \dots, b_r, \underbrace{1, \dots, 1}_s, \underbrace{0, \dots, 0}_{p-r-s})$  with  $b_1 \geq b_2 \geq \dots \geq b_r \geq 2$  and  $2r + s \leq p$ .*

Therefore, we may parametrize  $\sigma$  as

$$\begin{cases} (b_1, b_2, \dots, b_r, \underbrace{1, \dots, 1}_s, \underbrace{0, \dots, 0}_{[\frac{p}{2}] - r - s}; +1), & \text{if } r + s \leq \frac{p}{2}, \\ (b_1, b_2, \dots, b_r, \underbrace{1, \dots, 1}_{p-2r-s}, \underbrace{0, \dots, 0}_{[\frac{p}{2}] - p + r + s}; -1), & \text{if } \frac{p}{2} \leq r + s \leq p - r. \end{cases}$$

*Remark.* When  $r + s = \frac{p}{2}$ , the two cases coincide and give the same  $\sigma$ .

By this lemma  $O(p)$ -types are parametrized as  $\sigma = (a_1, a_2, \dots, a_x, 0, \dots, 0; \epsilon)$  with integers  $a_1 \geq a_2 \geq \dots \geq a_x \geq 1$ , and  $\epsilon \in \{\pm 1\}$ . The corresponding  $U(p)$ -type is

$$\lambda = (a_1, a_2, \dots, a_x, \underbrace{1, \dots, 1}_{\frac{1-\epsilon}{2}(p-2x)}, 0, \dots, 0).$$

When  $p$  is even and  $p = 2x$ ,  $\epsilon = \pm 1$  give the same  $\sigma$ . Let  $Id$  be the identity matrix in  $O(p)$ , then  $\sigma(-Id)$  acts by the scalar  $\epsilon^p \cdot (-1)^{\sum_{i=1}^m a_i}$ .

An  $O(p) \times O(q)$ -type is parametrized as  $(a_1, a_2, \dots, a_{[\frac{p}{2}]}; \epsilon) \otimes (b_1, b_2, \dots, b_{[\frac{q}{2}]}; \eta)$  with integers  $a_1 \geq a_2 \geq \dots \geq a_{[\frac{p}{2}]} \geq 0$ ,  $b_1 \geq b_2 \geq \dots \geq b_{[\frac{q}{2}]} \geq 0$ , and  $(\epsilon, \eta) \in \{\pm 1\} \times \{\pm 1\}$ . We refer to  $(a_1, a_2, \dots, a_{[\frac{p}{2}]}; b_1, b_2, \dots, b_{[\frac{q}{2}]})$  as its *highest weight*, and to  $(\epsilon, \eta)$  as its *signs*.

**2.4. Degree of  $K$ -types.** Consider the dual pair  $(G, G') = (O(p, q), Sp(2n, \mathbb{R}))$  with  $p + q$  even. In [How89] Howe defines a *degree* for each  $K$ -type  $\sigma$  for  $G$  or  $G'$  which occurs in the Fock space  $\mathcal{F}$  of  $\omega$ , which is the minimal degree of polynomials in the  $\sigma$ -isotypic subspace of  $\mathcal{F}$ . He also points out a correspondence of  $K$ -types in the space of joint harmonics  $\mathcal{H}$  (a certain subspace of  $\mathcal{F}$  associated to the dual pair). Let us describe the degrees and this correspondence explicitly.

**Proposition 4** ([Pau05, Proposition 4]). *Let  $\sigma = (a_1, a_2, \dots, a_x, 0, \dots, 0; \epsilon) \otimes (b_1, b_2, \dots, b_y, 0, \dots, 0; \eta)$  be a  $K$ -type for  $O(p, q)$ , with  $a_x \geq 1$  and  $b_y \geq 1$ . Let  $\sigma'$  be a  $K$ -type for  $Sp(2n, \mathbb{R})$  written as  $\sigma' = (\frac{p-q}{2}, \dots, \frac{p-q}{2}) + (c_1, c_2, \dots, c_n)$ . Let  $\mathcal{H}$  be the space of joint harmonics associated to the dual pair  $(O(p, q), Sp(2n, \mathbb{R}))$ . Let  $\mathcal{F}$  be the Fock space of  $\omega$  associated to this dual pair.*

(1) *The  $O(p) \times O(q)$ -type  $\sigma$  occurs in  $\mathcal{H}$  if and only if  $n \geq x + y + \frac{1-\epsilon}{2}(p-2x) + \frac{1-\eta}{2}(q-2y)$ . In that case  $\sigma$  corresponds to the  $U(n)$ -type*

$$\left(\frac{p-q}{2}, \dots, \frac{p-q}{2}\right) + (a_1, a_2, \dots, a_x, \underbrace{1, \dots, 1}_{\frac{1-\epsilon}{2}(p-2x)}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{\frac{1-\eta}{2}(q-2y)}, -b_y, \dots, -b_2, -b_1).$$

(2) *The  $U(n)$ -type  $\sigma'$  occurs in  $\mathcal{H}$  if and only if*

$$2\#\{i \mid c_i \geq 2\} + \#\{i \mid c_i = 1\} \leq p \quad \text{and} \quad 2\#\{i \mid c_i \leq -2\} + \#\{i \mid c_i = -1\} \leq q.$$

(3) *If  $\sigma$  occurs in  $\mathcal{F}$ , then its degree is  $\deg(\sigma) = \sum_{i=1}^x a_i + \sum_{i=1}^y b_i + \frac{1-\epsilon}{2}(p-2x) + \frac{1-\eta}{2}(q-2y)$ .*

(4) If  $\sigma'$  occurs in  $\mathcal{F}$ , then its degree is  $\deg(\sigma') = \sum_{i=1}^n |c_i|$ .

*Remark.* For our dual pairs, the degree of a  $K$ -type for  $O(p, q)$  is independent of  $n$ , and the degree of a  $K$ -type for  $Sp(2n, \mathbb{R})$  depends only on the difference  $p - q$ .

Write the correspondence of  $K$ -types in  $\mathcal{H}$  as

$$\{O(p) \times O(q)\text{-types in } \mathcal{H}\} \begin{array}{c} \xrightarrow{\phi_n} \\ \xleftarrow{\phi_{p,q}} \end{array} \{U(n)\text{-types in } \mathcal{H}\}.$$

For an  $O(p) \times O(q)$ -type  $\sigma$  and a  $U(n)$ -type  $\sigma'$ , if they correspond to each other in  $\mathcal{H}$ , write  $\phi_n(\sigma) = \sigma'$  and  $\phi_{p,q}(\sigma') = \sigma$ . If  $\sigma$  (resp.  $\sigma'$ ) does not occur in  $\mathcal{H}$ , write  $\phi_n(\sigma) = 0$  (resp.  $\phi_{p,q}(\sigma') = 0$ ).

Suppose that  $\pi \in \mathcal{R}(G)$  and  $\pi' \in \mathcal{R}(G')$  correspond to each other in the local theta correspondence. Let  $\mathcal{D}(\pi)$  denote the set of  $K$ -types for  $G$  which is of minimal degree in  $\pi$ , and similarly for  $\pi'$ . Define the *degree* of  $\pi$  as  $\deg(\pi) = \deg(\sigma)$  for any  $\sigma \in \mathcal{D}(\pi)$ , and similarly for  $\pi'$ .

**Lemma 5** ([How89]). *If  $\pi \in \mathcal{R}(G)$  and  $\theta_n(\pi) \neq 0$ , then  $\phi_n(\mathcal{D}(\pi)) = \mathcal{D}(\theta_n(\pi))$  and  $\deg(\pi) = \deg(\theta_n(\pi))$ .*

About the degrees of  $K$ -types, the following lemma is quite useful.

**Lemma 6.** *Suppose  $\pi \in \mathcal{R}(G)$  and  $\theta_n(\pi) \neq 0$ . If  $\pi$  contains two  $K$ -types  $\sigma_1$  and  $\sigma_2$ , then*

$$\deg(\sigma_1) \equiv \deg(\sigma_2) \pmod{2}.$$

*Proof.* Consider the  $(\mathfrak{g}, K)$ -module of  $\pi$ . Notice that  $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$ , where  $\mathcal{F}_i$  (for  $i \in \{0, 1\}$ ) is the linear span of all homogeneous polynomials in  $\mathcal{F}$  with degree  $\equiv i \pmod{2}$ . Each  $\mathcal{F}_i$  is  $(\mathfrak{g}, K)$ -invariant, so the  $(\mathfrak{g}, K)$ -module of  $\pi$  is equivalent to an irreducible  $(\mathfrak{g}, K)$ -quotient of either  $\mathcal{F}_0$  or  $\mathcal{F}_1$ . Thus one of  $\mathcal{F}_0$  and  $\mathcal{F}_1$  contains both  $\sigma_1$  and  $\sigma_2$ .

Suppose that  $\deg(\sigma_1)$  and  $\deg(\sigma_2)$  have different parity. Then one of  $\sigma_1$  and  $\sigma_2$ , denoted by  $\sigma$ , must occur in both  $\mathcal{F}_0$  and  $\mathcal{F}_1$ . Let  $\mathcal{F}(d)$  denote the subspace of homogeneous polynomials in  $\mathcal{F}$  with degree  $d$ . Since the action of  $K$  preserves the degree,  $\mathcal{F}(d)$  is  $K$ -invariant. Write  $\mathcal{F}_\sigma$  for the  $\sigma$ -isotypic subspace of  $\mathcal{F}$ . Then  $\mathcal{F}(d)_\sigma = \mathcal{F}_\sigma \cap \mathcal{F}(d)$  is the  $\sigma$ -isotypic subspace of  $\mathcal{F}(d)$ . For  $i \in \{0, 1\}$ , let

$$d_i = \min\{d \mid \mathcal{F}(d)_\sigma \neq 0, d \equiv i \pmod{2}\}$$

Recall that

$$\mathcal{H}(K) = \{P \in \mathcal{F} \mid l(P) = 0 \text{ for all } l \in \mathfrak{m}^{(0,2)}\},$$

where  $\mathfrak{m}^{(0,2)} = \mathfrak{m} \cap \mathfrak{sp}^{(0,2)}$ . Here  $\mathfrak{sp}$  is the complexified Lie algebra of  $\mathbf{Sp} = Sp(2n(p+q), \mathbb{R})$ ,  $\mathfrak{m}$  is the complexified Lie algebra of  $M =$  the centralizer of  $K$  in  $\mathbf{Sp}$ , and  $\mathfrak{sp}^{(0,2)}$  acts as the span of  $\frac{\partial^2}{\partial z_i \partial \bar{z}_j}$  (see [How89]). Since  $\mathfrak{m}$  and  $K$  commute,  $\mathcal{H}(K)$  is  $K$ -stable, and  $\mathcal{F}_\sigma$  is  $\mathfrak{m}$ -stable. The action of  $\mathfrak{sp}^{(0,2)}$  reduces the degree of polynomials by 2, so the action of  $\mathfrak{m}^{(0,2)}$  on  $\mathcal{F}(d_i)_\sigma$  gives results in  $\mathcal{F}_\sigma \cap \mathcal{F}(d_i - 2) = \mathcal{F}(d_i - 2)_\sigma = 0$ . Thus  $\mathcal{F}(d_i)_\sigma \subseteq \mathcal{H}(K)$  for both  $i \in \{0, 1\}$ . So  $\mathcal{F}(d_0)_\sigma \cup \mathcal{F}(d_1)_\sigma \subseteq \mathcal{H}(K)_\sigma$  and  $\mathcal{F}(d_0)_\sigma \cap \mathcal{F}(d_1)_\sigma = \emptyset$ . However, a result in classical invariant theory states that  $\mathcal{H}(K)_\sigma = \mathcal{F}(d)_\sigma$  for  $d = \deg(\sigma) = \min\{d_0, d_1\}$  ([How89, (3.9)(b)]). This makes a contradiction.  $\square$

**2.5. Lowest  $K$ -types.** Let  $\|\cdot\|$  be the norm of  $K$ -types defined by Vogan [Vog79]: for a  $K$ -type  $\sigma$ ,  $\|\sigma\| = \langle \sigma + 2\rho_c, \sigma + 2\rho_c \rangle$  where  $2\rho_c$  is the sum of all positive compact roots in  $\Delta_c^+$ . Indeed  $\Delta_c^+ = \Delta^+(\mathfrak{t}, \mathfrak{k})$ , which is a system of positive roots for a maximal torus  $T$  in  $K$ . So this definition depends only on the compact Lie group  $K$ .

**Proposition 7.** *If  $\sigma = (a_1, \dots, a_n)$  is a  $U(n)$ -type, then  $\|\sigma\| = \sum_{i=1}^n (a_i + n + 1 - 2i)^2$ .*

*Proof.* As  $\Delta_c^+ = \{e_i - e_j \mid 1 \leq i < j \leq n\}$ ,  $2\rho_c = \sum_{1 \leq i < j \leq n} (e_i - e_j) = \sum_{i=1}^n ((n-i) - (i-1))e_i$ . So  $\sigma + 2\rho_c = \sum_{i=1}^n (a_i + n + 1 - 2i)e_i$ , and  $\|\sigma\| = \sum_{i=1}^n (a_i + n + 1 - 2i)^2$ .  $\square$

**Proposition 8.** *If  $\sigma = (a_1, \dots, a_{\lfloor \frac{p}{2} \rfloor}; \epsilon) \otimes (b_1, \dots, b_{\lfloor \frac{q}{2} \rfloor}; \eta)$  is an  $O(p) \times O(q)$ -type with  $p+q$  even, then  $\|\sigma\| = \sum_{i=1}^{\lfloor \frac{p}{2} \rfloor} (a_i + p - 2i)^2 + \sum_{i=1}^{\lfloor \frac{q}{2} \rfloor} (b_i + q - 2i)^2$ .*

*Proof.* If  $p, q$  are even,

$$\begin{aligned} 2\rho_c &= \sum_{1 \leq i < j \leq \frac{p}{2}} ((e_i + e_j) + (e_i - e_j)) + \sum_{1 \leq i < j \leq \frac{q}{2}} ((f_i + f_j) + (f_i - f_j)) \\ &= \sum_{i=1}^{\frac{p}{2}} (p - 2i)e_i + \sum_{i=1}^{\frac{q}{2}} (q - 2i)f_i. \end{aligned}$$

If  $p, q$  are odd,

$$\begin{aligned} 2\rho_c &= \sum_{1 \leq i < j \leq \lfloor \frac{p}{2} \rfloor} ((e_i + e_j) + (e_i - e_j)) + \sum_{1 \leq i < j \leq \lfloor \frac{q}{2} \rfloor} ((f_i + f_j) + (f_i - f_j)) + \sum_{i=1}^{\lfloor \frac{p}{2} \rfloor} e_i + \sum_{i=1}^{\lfloor \frac{q}{2} \rfloor} f_i \\ &= \sum_{i=1}^{\lfloor \frac{p}{2} \rfloor} (p - 2i)e_i + \sum_{i=1}^{\lfloor \frac{q}{2} \rfloor} (q - 2i)f_i \end{aligned}$$

In both cases,  $\|\sigma\| = \langle \sigma + 2\rho_c, \sigma + 2\rho_c \rangle = \sum_{i=1}^{\lfloor \frac{p}{2} \rfloor} (a_i + p - 2i)^2 + \sum_{i=1}^{\lfloor \frac{q}{2} \rfloor} (b_i + q - 2i)^2$ .  $\square$

Let  $H$  be a reductive Lie group with a maximal compact subgroup  $K$ . Let  $\pi$  be an admissible representation of  $H$ . If  $\sigma$  occurs with minimal norm in  $\pi$ , we call  $\sigma$  a *lowest  $K$ -type* of  $\pi$ . Let  $\mathcal{A}(\pi)$  denote the set of all lowest  $K$ -types of  $\pi$ .

For  $\pi \in \mathcal{R}(Sp(2n, \mathbb{R}))$  or  $\mathcal{R}(O(p, q))$  with  $p+q$  even, there are explicit algorithms to calculate  $\mathcal{A}(\pi)$  from Langlands parameters in the following Subsection 2.6 and 2.7. These algorithms are Proposition 6, 10, and 13 of [Pau05]. To save space, we quote them in Appendix A, as Proposition 44, 45, and 46.

**2.6. Parametrization for  $\mathcal{R}(Sp(2n, \mathbb{R}))$ .** To “explicitly” describe or compute the local theta correspondence, we shall use some parametrization of the irreducible admissible representations. In this paper, we choose Vogan’s version of Langlands classification in [Vog84], explicitly described as in [Pau05]. Irreducible admissible representations of  $Sp(2n, \mathbb{R})$  are parametrized by *Langlands parameters*  $(\lambda_d, \Psi, \mu, \nu, \varepsilon, \kappa)$  as follows.

Let  $(W, \langle, \rangle)$  be a symplectic space over  $\mathbb{R}$  of dimension  $2n$  with isometry group  $Sp(W) \cong Sp(2n, \mathbb{R})$ . Let

$$\{0\} \subset W_1 \subset W_2 \subset \dots \subset W_r$$



be an isotropic flag in  $W$ , with  $\dim W_i = d_i$ . Set  $d_0 = 0$ , and  $n_i = d_i - d_{i-1}$ . Let  $P = MAN$  be the stabilizer of this flag in  $Sp(W)$ . Then  $P$  is a parabolic subgroup of  $Sp(W)$  with Levi factor

$$MA \cong Sp(2(n - d_r), \mathbb{R}) \times \prod_{i=1}^r GL(n_i, \mathbb{R}).$$

Especially, let  $(n_1, \dots, n_r) = (\underbrace{2, \dots, 2}_s, \underbrace{1, \dots, 1}_t)$  and  $n - d_r = n - 2s - t = v$ . Then

$$MA \cong Sp(2v, \mathbb{R}) \times GL(2, \mathbb{R})^s \times GL(1, \mathbb{R})^t.$$

Take  $\lambda_d \in \mathbb{Z}^v$ ,  $\mu \in (\mathbb{Z}_{\geq 0})^s$ ,  $\nu \in \mathbb{C}^s$ ,  $\varepsilon \in \{\pm 1\}^t$ , and  $\kappa \in \mathbb{C}^t$ , subject to the following conditions.

(1) Let  $(\lambda_d, \Psi)$  parametrize a limit of discrete series  $\rho = \rho(\lambda_d, \Psi)$  of  $Sp(2v, \mathbb{R})$ . Here  $\lambda_d$  is the Harish-Chandra parameter of  $\rho$ , of the form  $\lambda_d =$

$$(\underbrace{a_1, \dots, a_1}_{k_1}, \underbrace{a_2, \dots, a_2}_{k_2}, \dots, \underbrace{a_b, \dots, a_b}_{k_b}, \underbrace{0, \dots, 0}_z, \underbrace{-a_b, \dots, -a_b}_{l_b}, \dots, \underbrace{-a_2, \dots, -a_2}_{l_2}, \underbrace{-a_1, \dots, -a_1}_{l_1}),$$

with integers  $a_1 > a_2 > \dots > a_b > 0$ , and  $|k_i - l_i| \leq 1$  for all  $i$ . Furthermore,  $\Psi$  is a system of positive roots for  $Sp(2v, \mathbb{R})$  containing the standard set of positive compact roots  $\Delta_c^+ = \{e_i - e_j \mid 1 \leq i < j \leq v\}$ , such that  $\lambda_d$  is dominant with respect to  $\Psi$ , and satisfy the condition (F-1):

$$(F-1) \quad \text{if } \alpha \in \Delta_c^+ \text{ is a simple root in } \Psi, \text{ then } \langle \lambda_d, \alpha \rangle > 0.$$

(2) Let  $(\mu, \nu)$  parametrize a relative limit of discrete series

$$\tau = \tau(\mu, \nu) = \tau(\mu_1, \nu_1) \otimes \tau(\mu_2, \nu_2) \otimes \dots \otimes \tau(\mu_s, \nu_s)$$

of  $GL(2, \mathbb{R})^s$ , where  $\mu = (\mu_1, \mu_2, \dots, \mu_s) \in (\mathbb{Z}_{\geq 0})^s$ ,  $\nu = (\nu_1, \nu_2, \dots, \nu_s) \in \mathbb{C}^s$ , and  $\tau(\mu_i, \nu_i)$  is the relative limit of discrete series of  $GL(2, \mathbb{R})$  with infinitesimal character  $(\frac{1}{2}(\mu_i + \nu_i), \frac{1}{2}(-\mu_i + \nu_i))$  and lowest  $K$ -type  $(\mu_i + 1; 1)$ .

(3) Let  $(\varepsilon, \kappa)$  parametrize a character

$$\chi = \chi(\varepsilon, \kappa) = \chi(\varepsilon_1, \kappa_1) \otimes \chi(\varepsilon_2, \kappa_2) \otimes \dots \otimes \chi(\varepsilon_t, \kappa_t).$$

of  $GL(1, \mathbb{R})^t$ , where  $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_t) \in \{\pm 1\}^t$ ,  $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_t) \in \mathbb{C}^t$ , and  $\chi(\varepsilon_j, \kappa_j)$  is the character of  $GL(1, \mathbb{R})$  that maps  $x$  to  $\text{sgn}(x)^{\frac{1-\varepsilon_j}{2}} |x|^{\kappa_j}$ .

From above parameters, we obtain a standard module  $\text{Ind}_{MAN}^{Sp(2n, \mathbb{R})}(\rho \otimes \tau \otimes \chi \otimes \mathbb{1}_N)$ , where  $\mathbb{1}_N$  is the trivial representation of  $N$ . Here the induction is normalized so that infinitesimal characters are preserved.

**Lemma 9** ([Vog84, Pau05]). (1) If the parameters  $(\lambda_d, \Psi, \mu, \nu, \varepsilon, \kappa)$  are chosen as above, and satisfy the “non-parity condition (F-2)” for  $Sp(2n, \mathbb{R})$ :

$$\begin{aligned} & \text{for } 1 \leq i \leq s, \text{ if } \nu_i = 0, \text{ then } \mu_i \text{ is odd,} \\ & \text{for } 1 \leq i, j \leq t, \text{ if } \kappa_i = \pm \kappa_j, \text{ then } \varepsilon_i = \varepsilon_j, \\ & \text{for } 1 \leq i \leq t, \text{ if } \kappa_i = 0, \text{ then } \varepsilon_i = (-1)^v, \end{aligned}$$

then for a well chosen  $N$ , the standard module  $\text{Ind}_{MAN}^{Sp(2n, \mathbb{R})}(\rho \otimes \tau \otimes \chi \otimes \mathbb{1}_N)$  has a unique irreducible quotient, denoted by  $\pi(\lambda_d, \Psi, \mu, \nu, \varepsilon, \kappa)$ .

(2) Each irreducible admissible representation of  $Sp(2n, \mathbb{R})$  is infinitesimally equivalent to some  $\pi(\lambda_d, \Psi, \mu, \nu, \varepsilon, \kappa)$  obtained in (1).

(3)  $\pi(\lambda_d, \Psi, \mu, \nu, \varepsilon, \kappa)$  and  $\pi(\lambda'_d, \Psi', \mu', \nu', \varepsilon', \kappa')$  are infinitesimally equivalent if and only if:  $\lambda_d = \lambda'_d$ ,  $\Psi = \Psi'$ ,  $(\mu', \nu')$  is obtained from  $(\mu, \nu)$  by a simultaneous permutation of the coordinates



of  $\mu$  and  $\nu$ , and by possibly multiplying some of the entries of  $\nu$  by  $-1$ , and similarly  $(\varepsilon', \kappa')$  is obtained from  $(\varepsilon, \kappa)$  by a simultaneous permutation of the coordinates of  $\varepsilon$  and  $\kappa$ , and by possibly multiplying some of the entries of  $\kappa$  by  $-1$ .

*Remark.* Parameters  $\lambda_d, \mu, \nu, \varepsilon, \kappa$  that do not occur will be written as 0. To avoid confusion, each of them that occurs will be written in the form of  $(\cdot)$ , even if it has only one entry. Thus “ $\kappa = 0$ ” and “ $\kappa = (0)$ ” have totally different meanings.

**2.7. Parametrization for  $\mathcal{R}(O(p, q))$  with  $p + q$  even.** We describe the parametrization for  $\mathcal{R}(O(p, q))$  with  $p + q$  even in the same way as in [Pau05]. This parametrization is similar to that in the last subsection, but with two more parameters  $(\xi, \zeta)$  about the signs of lowest  $K$ -types. Irreducible admissible representations of  $O(p, q)$  (with  $p + q$  even) are parametrized by Langlands parameters  $(\zeta, \lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa)$  as follows.

Let  $(V, (\cdot, \cdot))$  be a real vector space with nondegenerate symmetric bilinear form  $(\cdot, \cdot)$  of signature  $(p, q)$ , with isometry group  $O(V) \cong O(p, q)$  (with  $p + q$  even). Let

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_r$$

be an isotropic flag in  $V$ , with  $\dim V_i = d_i$ . Set  $d_0 = 0$ , and  $n_i = d_i - d_{i-1}$ . Let  $P = MAN$  be the stabilizer of this flag in  $O(V)$ . Then  $P$  is a parabolic subgroup of  $O(V)$  with Levi factor

$$MA \cong O(p - d_r, q - d_r) \times \prod_{i=1}^r GL(n_i, \mathbb{R}).$$

Especially, let  $(n_1, \dots, n_r) = (\underbrace{2, \dots, 2}_s, \underbrace{1, \dots, 1}_t)$ , and let  $p - d_r = p - 2s - t$  be even. Write  $p - d_r = 2a$ ,  $q - d_r = 2d$  with  $a, d \in \mathbb{Z}$ . Then

$$MA \cong O(2a, 2d) \times GL(2, \mathbb{R})^s \times GL(1, \mathbb{R})^t.$$

Take  $\lambda_d \in \mathbb{Z}^{a+d}$ ,  $\mu \in (\mathbb{Z}_{\geq 0})^s$ ,  $\nu \in \mathbb{C}^s$ ,  $\varepsilon \in \{\pm 1\}^t$ ,  $\kappa \in \mathbb{C}^t$ ,  $\xi \in \{\pm 1\}$ , and  $\zeta \in \{\pm 1\}$ , subject to the following conditions.

(1) Let  $(\lambda_d, \xi, \Psi)$  parametrize a limit of discrete series  $\rho = \rho(\lambda_d, \xi, \Psi)$  of  $O(2a, 2d)$ . Here  $\lambda_d$  is the Harish-Chandra parameter of  $\rho$ , of the form  $\lambda_d =$

$$(\underbrace{a_1, \dots, a_1}_{k_1}, \underbrace{a_2, \dots, a_2}_{k_2}, \dots, \underbrace{a_b, \dots, a_b}_{k_b}, \underbrace{0, \dots, 0}_z; \underbrace{a_1, \dots, a_1}_{l_1}, \underbrace{a_2, \dots, a_2}_{l_2}, \dots, \underbrace{a_b, \dots, a_b}_{l_b}, \underbrace{0, \dots, 0}_{z'}),$$

with integers  $a_1 > a_2 > \cdots > a_b > 0$ ,  $|k_i - l_i| \leq 1$  for all  $i$ , and  $|z - z'| \leq 1$ . Furthermore,  $\Psi$  is a system of positive roots for  $O(2a, 2d)$  containing the standard set of positive compact roots

$$\Delta_c^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq a\} \cup \{f_i \pm f_j \mid 1 \leq i < j \leq d\}$$

such that  $\lambda_d$  is dominant with respect to  $\Psi$ , and satisfy the condition (F-1):

$$(F-1) \quad \text{if } \alpha \in \Delta_c^+ \text{ is a simple root in } \Psi, \text{ then } \langle \lambda_d, \alpha \rangle > 0.$$

Indeed, the Harish-Chandra parameter  $\lambda_d$  and positive root system  $\Psi$  determine a limit of discrete series of  $SO(2a, 2d)$  denoted by  $\rho(\lambda_d, \Psi)$ . When  $z + z' = 0$ ,  $\text{Ind}_{SO(2a, 2d)}^{O(2a, 2d)} \rho(\lambda_d, \Psi)$  is irreducible, and thus is a limit of discrete series of  $O(2a, 2d)$  denoted by  $\rho(\lambda_d, 1, \Psi)$ . (So  $\xi \neq -1$  in this case). When  $z + z' > 0$ ,  $\text{Ind}_{SO(2a, 2d)}^{O(2a, 2d)} \rho(\lambda_d, \Psi)$  is the direct sum of two limits of discrete series of  $O(2a, 2d)$ , precisely one of which has the lowest  $K$ -type with signs  $(1; 1)$ , denoted by  $\rho(\lambda_d, 1, \Psi)$ , and the other one by  $\rho(\lambda_d, -1, \Psi)$ .

(2) Let  $(\mu, \nu)$  parametrize a relative limit of discrete series

$$\tau = \tau(\mu, \nu) = \tau(\mu_1, \nu_1) \otimes \tau(\mu_2, \nu_2) \otimes \cdots \otimes \tau(\mu_s, \nu_s)$$

of  $GL(2, \mathbb{R})^s$ , where  $\mu = (\mu_1, \mu_2, \dots, \mu_s) \in (\mathbb{Z}_{\geq 0})^s$ ,  $\nu = (\nu_1, \nu_2, \dots, \nu_s) \in \mathbb{C}^s$ , and  $\tau(\mu_i, \nu_i)$  is the relative limit of discrete series of  $GL(2, \mathbb{R})$  with infinitesimal character  $(\frac{1}{2}(\mu_i + \nu_i), \frac{1}{2}(-\mu_i + \nu_i))$  and lowest  $K$ -type  $(\mu_i + 1; 1)$ .

(3) Let  $(\varepsilon, \kappa)$  parametrize a character

$$\chi = \chi(\varepsilon, \kappa) = \chi(\varepsilon_1, \kappa_1) \otimes \chi(\varepsilon_2, \kappa_2) \otimes \cdots \otimes \chi(\varepsilon_t, \kappa_t).$$

of  $GL(1, \mathbb{R})^t$ , where  $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_t) \in \{\pm 1\}^t$ ,  $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_t) \in \mathbb{C}^t$ , and  $\chi(\varepsilon_j, \kappa_j)$  is the character of  $GL(1, \mathbb{R})$  that maps  $x$  to  $\text{sgn}(x)^{\frac{1-\varepsilon_j}{2}} |x|^{\kappa_j}$ .

From above parameters, we obtain a standard module  $\text{Ind}_{MAN}^{O(p,q)}(\rho \otimes \tau \otimes \chi \otimes \mathbb{1}_N)$  with normalized induction, where  $\mathbb{1}_N$  is the trivial representation of  $N$ .

**Lemma 10** ([Vog84, Pau05]). (1) Let the parameters  $(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa)$  be as above, and satisfy the “non-parity condition (F-2)” for  $O(p, q)$  with  $p + q$  even:

$$\begin{aligned} &\text{for } 1 \leq i \leq s, \text{ if } \nu_i = 0, \text{ then } \mu_i \text{ is odd,} \\ &\text{for } 1 \leq i, j \leq t, \text{ if } \kappa_i = \pm \kappa_j, \text{ then } \varepsilon_i = \varepsilon_j. \end{aligned}$$

Then for a well chosen  $N$ , the standard module  $\text{Ind}_{MAN}^{O(p,q)}(\rho \otimes \tau \otimes \chi \otimes \mathbb{1}_N)$  has a unique irreducible quotient, denoted by  $\pi_1(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa)$ , if  $\lambda_d$  contains a zero entry or  $\kappa$  contains no zero entry. Otherwise, we get two irreducible quotients distinguished by the signs of their lowest  $K$ -types (as described in Proposition 46), denoted by  $\pi_1(\lambda_d, 1, \Psi, \mu, \nu, \varepsilon, \kappa)$  and  $\pi_{-1}(\lambda_d, 1, \Psi, \mu, \nu, \varepsilon, \kappa)$  respectively.

(2) Each irreducible admissible representation of  $O(p, q)$  is infinitesimally equivalent to some  $\pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa)$  obtained in (1).

(3) Two such representations  $\pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa)$  and  $\pi_{\zeta'}(\lambda'_d, \xi', \Psi', \mu', \nu', \varepsilon', \kappa')$  are infinitesimally equivalent if and only if:  $\lambda_d = \lambda'_d$ ,  $\Psi = \Psi'$ ,  $\xi = \xi'$ ,  $\zeta = \zeta'$ ,  $(\mu', \nu')$  is obtained from  $(\mu, \nu)$  by a simultaneous permutation of the coordinates of  $\mu$  and  $\nu$ , and by possibly multiplying some of the entries of  $\nu$  by  $-1$ , and similarly  $(\varepsilon', \kappa')$  is obtained from  $(\varepsilon, \kappa)$  a simultaneous permutation of the coordinates of  $\varepsilon$  and  $\kappa$ , and by possibly multiplying some of the entries of  $\kappa$  by  $-1$ .

*Remark.*  $\zeta = -1 \Rightarrow \lambda_d$  contains no zero entry  $\Rightarrow \xi = 1$ . Therefore,  $(\xi, \zeta) \neq (-1, -1)$ .

**2.8. Infinitesimal character.** Let  $G$  be a reductive Lie group with complexified Lie algebra  $\mathfrak{g}$ , and  $Z(\mathfrak{g})$  the center of the universal enveloping algebra  $U(\mathfrak{g})$ . For  $\pi \in \mathcal{R}(G)$  and  $z \in Z(\mathfrak{g})$ , Schur’s Lemma shows that  $\pi(z)$  acts by a scalar  $\lambda(z) \in \mathbb{C}$ . Then  $z \mapsto \lambda(z)$  gives a homomorphism of algebras  $\lambda : Z(\mathfrak{g}) \rightarrow \mathbb{C}$ , which is called the infinitesimal character of  $\pi$ .

Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{g}$ . Let  $W$  be the Weyl group  $N_G(\mathfrak{h})/Z_G(\mathfrak{h})$ . Via the Harish-Chandra isomorphism  $Z(\mathfrak{g}) \simeq S(\mathfrak{h})^W$  (the set of  $W$ -fixed elements of the symmetric algebra  $S(\mathfrak{h})$ ), the infinitesimal character of  $\pi$  is written as an element of  $\mathfrak{h}^*$ , up to the action of  $W$ .

**Proposition 11** ([Pau05]). The infinitesimal character of  $\pi(\lambda_d, \Psi, \mu, \nu, \varepsilon, \kappa) \in \mathcal{R}(Sp(2n, \mathbb{R}))$  or  $\pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa) \in \mathcal{R}(O(p, q))$  with  $p + q$  even is

$$(\lambda_d \mid (\frac{\mu_1 + \nu_1}{2}, \frac{\mu_2 + \nu_2}{2}, \dots, \frac{\mu_s + \nu_s}{2}, \frac{-\mu_1 + \nu_1}{2}, \frac{-\mu_2 + \nu_2}{2}, \dots, \frac{-\mu_s + \nu_s}{2}, \kappa_1, \dots, \kappa_t)),$$

up to the action of  $W$  which consists of all permutations and sign-changes of coordinates.

[Prz96] gives the duality correspondence of infinitesimal characters under the local theta correspondence for all reductive dual pairs over archimedean fields.

**Proposition 12** ([Prz96]). *Let  $\pi \in \mathcal{R}(O(p, q))$  (with  $p + q$  even) and  $\pi' \in \mathcal{R}(Sp(2n, \mathbb{R}))$  correspond in the local theta correspondence. Write the infinitesimal character of  $\pi$  as  $x = (x_1, \dots, x_m) \in \mathbb{C}^m$  with  $m = \frac{p+q}{2}$ , and that of  $\pi'$  as  $y = (y_1, \dots, y_n) \in \mathbb{C}^n$ . Let  $\sim$  denote the equivalence up to permutations and sign-changes of coordinates.*

- (1) *If  $m = n$ , then  $x \sim y$ .*
- (2) *If  $m > n$ , then  $x \sim (y \mid (0, 1, 2, \dots, m - n - 1))$ .*
- (3) *If  $m < n$ , then  $y \sim (x \mid (1, 2, 3, \dots, n - m))$ .*

### 3. EXPLICIT INDUCTION PRINCIPLE

Let us formulate the *explicit induction principle* for  $(O(p, q), Sp(2n, \mathbb{R}))$  with  $p + q$  even.

**3.1. Explicit induction principle on  $n$ .** Let  $\pi \in \mathcal{R}(O(p, q))$  (with  $p + q$  even) and  $\theta_n(\pi) = \pi' \in \mathcal{R}(Sp(2n, \mathbb{R}))$ . Similarly as in Subsection 2.6, let  $(W', \langle, \rangle)$  be a symplectic space over  $\mathbb{R}$  of dimension  $2(n + 1)$  with isometry group  $Sp(W') \cong Sp(2(n + 1), \mathbb{R})$ . Take an isotropic subspace  $W_0$  of dimension 1 in  $W'$ . Then the stabilizer of  $W_0$  in  $Sp(W')$  is a parabolic subgroup  $M'A'N'$  of  $Sp(2(n + 1), \mathbb{R})$  with Levi factor

$$M'A' \cong Sp(2n, \mathbb{R}) \times GL(1, \mathbb{R}).$$

**Lemma 13** ([Pau05, Theorem 30(1)]). *There exists a nontrivial  $O(p, q) \times Sp(2(n + 1), \mathbb{R})$ -map (on the level of Harish-Chandra modules)*

$$\omega \longrightarrow \pi \otimes \text{Ind}_{M'A'N'}^{Sp(2(n+1), \mathbb{R})}(\pi' \otimes \xi' \otimes \mathbb{1}_{N'}),$$

where  $\xi'$  is the character of  $GL(1, \mathbb{R})$  given by  $\xi'(g) = |\det(g)|^{\frac{p+q}{2}-n-1} \text{sgn}(\det(g))^{\frac{p-q}{2}}$ , and  $\mathbb{1}_{N'}$  is the trivial representation of  $N'$ . Let  $I'$  denote the above normalized induction. Then  $\theta_{n+1}(\pi)$  is an irreducible subquotient of  $I'$ .

Write  $\theta_n(\pi) = \pi' = \pi(\lambda_d, \Psi, \mu, \nu, \varepsilon, \kappa)$ . Let  $\mathbf{I}' = \mathbf{I}(\lambda_d, \Psi, \mu, \nu, (\varepsilon|((-1)^{\frac{p-q}{2}})), (\kappa|(1+n-\frac{p+q}{2})))$  be the standard module of parabolic induction of  $Sp(2(n+1), \mathbb{R})$  with the given Langlands parameters. Then  $I'$  is a subquotient of  $\mathbf{I}'$ , and  $\theta_{n+1}(\pi)$  is an irreducible subquotient of  $\mathbf{I}'$ .

It is well-known that in a standard modules of parabolic induction, each lowest  $K$ -type occurs with multiplicity one (cf. [Vog79]). Therefore, if  $\tau \in \mathcal{A}(\mathbf{I}')$ , then there is a unique irreducible subquotient of  $\mathbf{I}'$  containing  $\tau$ .

When  $p + q \neq 2n + 2$ , let  $\pi'_1 = \pi(\lambda_d, \Psi, \mu, \nu, (\varepsilon|((-1)^{\frac{p-q}{2}})), (\kappa|(1+n-\frac{p+q}{2})))$  with a possible modification on parameters: if  $\pm\kappa_i = 1 + n - \frac{p+q}{2}$  and  $\varepsilon_i \neq (-1)^{\frac{p-q}{2}}$  for some  $i$ , delete these four entries, and add  $(\mu_{s+1}, \nu_{s+1}) = (0, 2\kappa_i)$  into  $(\mu, \nu) = ((\mu_1, \dots, \mu_s), (\nu_1, \dots, \nu_s))$ . Since  $1 + n - \frac{p+q}{2} \neq 0$ , no zero entry is added. So the resulting parameters satisfy the condition (F-2). Whether modified or not,  $\pi'_1$  is infinitesimal equivalent to an irreducible subquotient of  $\mathbf{I}'$  (see [Vog81, Vog84]).

**Proposition 14.** *Suppose  $p + q \neq 2n + 2$ , if  $\theta_{n+1}(\pi)$  contains a lowest  $K$ -type of  $\pi'_1$ , then  $\theta_{n+1}(\pi) = \pi'_1$ .*

*Proof.* The  $K$ -structure of a standard module depends only on the discrete Langlands parameters. So  $\mathcal{A}(\mathbf{I}')$  can be calculated from the parameters  $(\lambda_d, \Psi, \mu, \nu, (\varepsilon|((-1)^{\frac{p-q}{2}})), (\kappa|(1+n-\frac{p+q}{2})))$  by the algorithm in Proposition 44. For  $\pi'_1$  this algorithm outputs the same result, so  $\mathcal{A}(\mathbf{I}') = \mathcal{A}(\pi'_1)$ . Both  $\theta_{n+1}(\pi)$  and  $\pi'_1$  are infinitesimal equivalent to irreducible subquotients of  $\mathbf{I}'$ , and

they contains the same lowest  $K$ -type of  $\mathbf{I}'$  which occurs with multiplicity one, so they are infinitesimal equivalent to each other.  $\square$

We try to find a sufficient condition to make  $\theta_{n+1}(\pi)$  contain a lowest  $K$ -type of  $\pi'_1$ . On the one hand,  $\theta_{n+1}(\pi)$  contains  $K$ -types in  $\mathcal{D}(\theta_{n+1}(\pi)) = \phi_{n+1}(\phi_{p,q}(\mathcal{D}(\pi')))$ . On the other hand,  $\mathcal{A}(\pi'_1)$  can be got from  $\mathcal{A}(\pi')$  by a simple operation when  $\pi'$  satisfy certain conditions. If  $\mathcal{D}(\pi') \cap \mathcal{A}(\pi')$  is nonempty, and  $\phi_{n+1} \circ \phi_{p,q}$  coincides with this operation on lowest  $K$ -types, then it is done.

**Proposition 15.** *Suppose  $\sigma'$  is a  $U(n)$ -type. If  $\sigma = \phi_{p,q}(\sigma') \neq 0$ , then  $\phi_{n+1}(\sigma) = \sigma'_1$ , where  $\sigma'_1$  is the  $U(n+1)$ -type parametrized by the same entries of  $\sigma'$  with one more entry  $\frac{p-q}{2}$  added.*

*Proof.* Check by the explicit descriptions of  $\phi_n$  and  $\phi_{n+1}$  in Proposition 4.  $\square$

Let  $k(\lambda_d)$  be the number of positive entries in  $\lambda_d$ ,  $l(\lambda_d)$  negative ones, and  $z(\lambda_d)$  zero ones.

**Proposition 16.** *Suppose  $(\lambda_d, \Psi)$  satisfy one of the following conditions:*

- (1)  $\frac{p-q}{2} = k(\lambda_d) - l(\lambda_d)$ ;
- (2)  $\frac{p-q}{2} = k(\lambda_d) - l(\lambda_d) + 1$ ,  $z(\lambda_d) > 0$ , and  $e_{k(\lambda_d)+1} + e_{k(\lambda_d)+z(\lambda_d)} \in \Psi$ ;
- (3)  $\frac{p-q}{2} = k(\lambda_d) - l(\lambda_d) - 1$ ,  $z(\lambda_d) > 0$ , and  $e_{k(\lambda_d)+1} + e_{k(\lambda_d)+z(\lambda_d)} \notin \Psi$ ;

*Then  $\mathcal{A}(\pi'_1) = \{\sigma'_1 \mid \sigma' \in \mathcal{A}(\pi')\}$ , where  $\sigma'_1$  has entries of  $\sigma'$  with one more entry  $\frac{p-q}{2}$  added.*

*Proof.* Compare the algorithms (Theorem 44) to compute  $\mathcal{A}(\pi')$  and  $\mathcal{A}(\pi'_1)$ , the only change is to add one more entry. Notice that  $k(\lambda_d) - l(\lambda_d) = u - r$  in the algorithm. The added entry is  $u - r$  in case (1),  $u - r + 1$  in case (2),  $u - r - 1$  in case (3), which is always  $\frac{p-q}{2}$ .  $\square$

**Lemma 17.** *Let  $\pi' = \pi(\lambda_d, \Psi, \mu, \nu, \varepsilon, \kappa) \in \mathcal{R}(Sp(2n, \mathbb{R}))$ . Let  $p$  and  $q$  be nonnegative integers with  $p + q$  even. Consider the following conditions.*

- (i)  $\theta_{p,q}(\pi') \neq 0$ , and  $p + q \leq 2n$ .
- (ii)  $\theta_{p',q'}(\pi') \neq 0$  for  $p' = p + n - \frac{p+q}{2}$  and  $q' = q + n - \frac{p+q}{2}$ .
- (iii)  $(\lambda_d, \Psi)$  satisfy the hypothesis of Proposition 16 (concerning only  $p - q$ ).
- (iv)  $\mathcal{A}(\pi') \subseteq \mathcal{D}(\pi')$  for the dual pair  $(O(p, q), Sp(2n, \mathbb{R}))$ .

*Then (i)  $\Rightarrow$  (ii)  $\Leftrightarrow$  (iii)  $\Rightarrow$  (iv).*

*Proof.* (i)  $\Rightarrow$  (ii): By Kudla's Persistence Principle as  $n - \frac{p+q}{2} \geq 0$ .

(ii)  $\Leftrightarrow$  (iii) From [Pau05, Theorem 18] we can read off the full description of  $\theta_{p',q'}$  for  $p' + q' = 2n$ , and see that  $\theta_{p',q'}(\pi') \neq 0$  if and only if  $(\lambda_d, \Psi)$  satisfy the hypothesis of Proposition 16.

(ii)  $\Rightarrow$  (iv) Notice that the degree of a  $K$ -type for  $Sp(2n, \mathbb{R})$  depends only on the difference  $p' - q' = p - q$  (by Proposition 4). By [Pau05, Corollary 37],  $\mathcal{A}(\pi') \subseteq \mathcal{D}(\pi')$ .  $\square$

**Proposition 18.** *If  $p + q \neq 2n + 2$  and  $(\lambda_d, \Psi)$  satisfy the hypothesis of Proposition 16, then  $\theta_{n+1}(\pi) = \pi'_1$ .*

*Proof.* By the last lemma, take any  $\sigma \in \mathcal{A}(\pi') \subseteq \mathcal{D}(\pi')$ , and we get  $\sigma'_1 \in \mathcal{A}(\pi'_1) \cap \mathcal{D}(\theta_{n+1}(\pi))$ .  $\theta_{n+1}(\pi)$  contains a lowest  $K$ -type of  $\pi'_1$ , so  $\theta_{n+1}(\pi) = \pi'_1$  by Proposition 14.  $\square$

**Theorem 19** (Explicit induction principle on  $n$ ). *Let  $\pi' = \pi(\lambda_d, \Psi, \mu, \nu, \varepsilon, \kappa) \in \mathcal{R}(Sp(2n, \mathbb{R}))$ ,  $\theta_{p,q}(\pi') \neq 0$  with  $p+q$  even and  $p+q \neq 2n+2$ . Suppose that  $\pi'$  satisfy the hypothesis of Proposition 16 (which is automatically true if  $p+q \leq 2n$  by Lemma 17). Then for any positive integer  $k$  such that  $\frac{p+q}{2} \notin [n+1, n+k]$  (if  $p+q \leq 2n$ , then for all  $k \geq 1$ ),*

$$\theta_{n+k}(\theta_{p,q}(\pi')) = \pi(\lambda_d, \Psi, \mu, \nu, (\varepsilon|((-1)^{\frac{p-q}{2}}, (-1)^{\frac{p-q}{2}}, \dots, (-1)^{\frac{p-q}{2}})), \\ (\kappa|(1+n-\frac{p+q}{2}, 2+n-\frac{p+q}{2}, \dots, k+n-\frac{p+q}{2})))$$

with a possible modification: if the resulting parameters contain entries  $\varepsilon_i \neq \varepsilon_j$  and  $\kappa_i = \pm\kappa_j$ , delete  $\varepsilon_i, \varepsilon_j, \kappa_i, \kappa_j$  from  $(\varepsilon, \kappa)$ , and add entries  $(0, 2\kappa_i)$  into  $(\mu, \nu)$ .

*Proof.* Use Corollary 18 repeatedly for  $k$  times.  $\square$

For  $\pi \in \mathcal{R}(O(p, q))$  with  $p+q$  even, define the *first occurrence index*

$$n(\pi) = \min\{0 \leq k \in \mathbb{Z} \mid \theta_k(\pi) \neq 0\}.$$

By the explicit induction principle on  $n$ , the question to compute the Langlands parameters of  $\theta_n(\pi)$  for all  $n \geq \frac{p+q}{2}$  reduce to the case when  $n = \max\{n(\pi), \frac{p+q}{2}\}$ . By the nonvanishing of the stable range theta liftings,  $n(\pi) \leq p+q$ . Therefore, to describe  $\theta_n$  for all  $n$ , we only need to consider for  $1 \leq n \leq p+q$ .

**3.2. Explicit induction principle on  $(p, q)$ .** Let  $\pi' \in \mathcal{R}(Sp(2n, \mathbb{R}))$  and  $\theta_{p,q}(\pi') = \pi \in \mathcal{R}(O(p, q))$  with  $p+q$  even. Similarly as in Subsection 2.7, let  $(V', (, ))$  be a real vector space with nondegenerate symmetric bilinear form  $(, )$  of signature  $(p+1, q+1)$  with isometry group  $O(V') \cong O(p+1, q+1)$ . Take an isotropic subspace  $V_0$  of dimension 1 in  $V'$ . Then the stabilizer of  $V_0$  in  $O(V')$  is a parabolic subgroup  $M'A'N'$  of  $O(p+1, q+1)$  with Levi factor

$$M'A' \cong O(p, q) \times GL(1, \mathbb{R}).$$

**Lemma 20** ([Pau05, Theorem 30(2)]). *There exists a nontrivial  $O(p+1, q+1) \times Sp(2n, \mathbb{R})$ -map (on the level Harish-Chandra modules)*

$$\omega \longrightarrow \text{Ind}_{M'A'N'}^{O(p+1, q+1)}(\pi \otimes \xi \otimes \mathbb{1}_{N'}) \otimes \pi',$$

where  $\xi$  is the character of  $GL(1, \mathbb{R})$  given by  $\xi'(g) = |\det(g)|^{n-\frac{p+q}{2}}$ , and  $\mathbb{1}_{N'}$  is the trivial representation of  $N'$ . Let  $I$  denote the above normalized induction. Then  $\theta_{p+1, q+1}(\pi')$  is an irreducible subquotient of  $I$ .

Write  $\theta_{p,q}(\pi') = \pi = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa)$ . Let  $\mathbf{I} = \mathbf{I}(\lambda_d, \xi, \Psi, \mu, \nu, (\varepsilon|(1)), (\kappa|(n - \frac{p+q}{2})))$  be the standard module of parabolic induction of  $O(p+1, q+1)$  with the given Langlands parameters. Then  $I$  is a subquotient of  $\mathbf{I}$ , and  $\theta_{p+1, q+1}(\pi')$  is an irreducible subquotient of  $\mathbf{I}$ .

When  $p+q \neq 2n$ , let  $\pi_{1,1} = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, (\varepsilon|(1)), (\kappa|(n - \frac{p+q}{2})))$  with a possible modification: if  $\pm\kappa_i = n - \frac{p+q}{2}$  and  $\varepsilon_i \neq 1$  for some  $i$ , delete these four entries, and add  $(\mu_{s+1}, \nu_{s+1}) = (0, 2\kappa_i)$  into  $(\mu, \nu) = ((\mu_1, \dots, \mu_s), (\nu_1, \dots, \nu_s))$ . Since  $n - \frac{p+q}{2} \neq 0$ , no zero entry is added. So the resulting parameters satisfy the condition (F-2). Whether modified or not,  $\pi_{1,1}$  is equivalent to an irreducible subquotient of  $\mathbf{I}$  (see [Vog81, Vog84]).

**Proposition 21.** *Suppose  $p+q \neq 2n$ . If  $\theta_{p+1, q+1}(\pi')$  contains a lowest  $K$ -type of  $\pi_{1,1}$ , then  $\theta_{p+1, q+1}(\pi') = \pi_{1,1}$ .*

*Proof.* The  $K$ -structure of a standard module depends only on the discrete Langlands parameters. So  $\mathcal{A}(\mathbf{I})$  can be calculated from the parameters  $(\lambda_d, \xi, \Psi, \mu, \nu, (\varepsilon|(1)), (\kappa|(n - \frac{p+q}{2})))$  by the algorithm in Proposition 45 and 46 (with all the signs obtained from all choices of  $\zeta$ ). Comparing to the algorithm to compute  $\mathcal{A}(\pi_{1,1})$ , we see  $\mathcal{A}(\pi_{1,1}) \subseteq \mathcal{A}(\mathbf{I})$ . Both  $\theta_{p+1,q+1}(\pi')$  and  $\pi_{1,1}$  are equivalent to irreducible subquotients of  $\mathbf{I}$ , and they contains the same lowest  $K$ -type of  $\mathbf{I}$  which occurs with multiplicity one, so they are equivalent to each other.  $\square$

**Proposition 22.** *Suppose  $\sigma = (a_1, \dots, a_x, 0, \dots, 0; \epsilon) \otimes (b_1, \dots, b_y, 0, \dots, 0; \eta)$  is an  $O(p) \times O(q)$ -type, with  $a_x > 0$  and  $b_y > 0$ . (When  $p = 2x$ ,  $\epsilon = \pm 1$  give the same  $O(p)$ -type, but we choose  $\epsilon = 1$  for the convenience. We write  $\epsilon = -1$  only if  $p > 2x$ . Similarly, we write  $\eta = -1$  only if  $q > 2y$ .) If  $\sigma' = \phi_n(\sigma) \neq 0$ . Then  $\phi_{p+1,q+1}(\sigma') = \sigma_{1,1}$ , where  $\sigma_{1,1}$  is defined as*

$$\sigma_{1,1} = (a_1, \dots, a_x, \frac{1-\epsilon}{2}, \underbrace{0, \dots, 0}_{[\frac{p+1}{2}-x-1}; \epsilon) \otimes (b_1, \dots, b_y, \frac{1-\eta}{2}, \underbrace{0, \dots, 0}_{[\frac{q+1}{2}-y-1}; \eta).$$

*Remark.* Notice that when  $p = 2x + 1$  and  $\epsilon = -1$ , the resulting  $(a_1, \dots, a_x, 1; -1)$  should be rewritten as  $(a_1, \dots, a_x, 1; 1)$  if we want to repeat this algorithm to get  $\phi_{p+2,q+2}(\sigma')$ . Similarly for  $q = 2y + 1$  and  $\eta = -1$ .

**Proposition 23.** *Suppose that  $p+q \neq 2n$ ,  $\pi = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa)$ , and  $\pi_{1,1}$  is as above. Also suppose that  $\zeta = \xi = 1$ , and either  $\lambda_d$  contains a zero entry or some  $(\varepsilon_i, \kappa_i) = (1, 0)$ . Then for any  $\sigma \in \mathcal{A}(\pi)$ , we have  $\sigma_{1,1} \in \mathcal{A}(\pi_{1,1})$ , where  $\sigma_{1,1}$  is got from  $\sigma$  by the above algorithm.*

*Proof.* Let  $\beta$  and  $\gamma$  be the numbers of indices  $j$  such that  $\varepsilon_j = 1$  and  $\varepsilon_j = -1$  respectively. For every  $\sigma \in \mathcal{A}(\pi)$ , under the assumptions for parameters of  $\pi$ , the signs of  $\sigma$  is determined by  $\beta$ ,  $\gamma$  and the highest weight  $(\Lambda_1; \Lambda_2)$  of  $\sigma$  as in Proposition 46 in Appendix A:

If  $\beta \geq \gamma$ , then the signs of  $\sigma$  are  $(1, 1)$ .

If  $\beta < \gamma$ , then the signs of  $\sigma$  are  $(1; -1)$  if  $\Lambda_1$  has more zeros than  $\Lambda_2$ , and  $(-1; 1)$  otherwise.

(Notice that when  $\beta < \gamma$ ,  $\Lambda_1$  or  $\Lambda_2$  contains no zero entry  $\Leftrightarrow \beta + 1 = \gamma \Rightarrow p$  and  $q$  are odd. So the signs of  $\sigma$  are well written as in the Proposition 22 to get  $\sigma_{1,1}$ .)

Compare the algorithms (Proposition 45, 46 in Appendix A) to get  $\mathcal{A}(\pi)$  and  $\mathcal{A}(\pi_{1,1})$ . The only change is from  $\beta$  to  $\beta + 1$ . If  $\beta \geq \gamma$  for  $\sigma$ , we get a lowest  $K$ -type of  $\pi_{1,1}$  with the same nonzero entries. If  $\beta < \gamma$  for  $\sigma$ , we get a lowest  $K$ -type of  $\pi_{1,1}$  with the same nonzero entries, and one more entry 1 in the left or right part that contains less zero entries. Always the resulting lowest  $K$ -type of  $\pi_{1,1}$  is  $\sigma_{1,1}$ .

(Notice that if  $\beta + 1 = \gamma$ , then the the resulting lowest  $K$ -type has signs  $(1, 1)$ , which coincides with the remark after Proposition 22 to get  $\sigma_{1,1}$ .)  $\square$

**Lemma 24.** *Suppose  $\pi = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa) \in \mathcal{R}(O(p, q))$  with  $p + q$  even. Consider the following conditions.*

(i)  $n(\pi) \leq \frac{p+q}{2} - 1$ .

(ii)  $\zeta = \xi = 1$ , and either  $\lambda_d$  contains a zero entry or some  $(\varepsilon_i, \kappa_i) = (1, 0)$ .

(iii)  $\mathcal{A}(\pi) \subseteq \mathcal{D}(\pi)$ .

Then (i)  $\Leftrightarrow$  (ii)  $\Rightarrow$  (iii).

*Proof.* (i)  $\Rightarrow$  (ii): Take  $n = \frac{p+q}{2} - 1 \geq n(\pi)$ . Then  $\theta_n(\pi) \neq 0$  by Kudla's Persistence Principle. So (ii) can read off from the complete description of  $\theta_n$  with  $2n+2 = p+q$  in [Pau05, Theorem 15,18].

(ii)  $\Rightarrow$  (i): [Pau05, Theorem 15,18] explicitly gives nonzero  $\theta_{\frac{p+q}{2}-1}(\pi)$  when (ii) holds.

(i) and (ii)  $\Rightarrow$  (iii): By [Pau05, Corollary 37], (i) implies that for any  $\sigma \in \mathcal{A}(\pi)$ , there is  $\delta \in \mathcal{A}(\pi) \cap \mathcal{D}(\pi)$  with the same highest weight as that of  $\sigma$ . However, (ii) implies that the choice for signs of lowest  $K$ -types of  $\pi$  with given highest weight is unique (see Proposition 46). So  $\delta = \sigma$ .  $\square$

**Theorem 25** (Explicit induction principle on  $(p, q)$ ). *Let  $\pi = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa) \in \mathcal{R}(O(p, q))$  with  $p+q$  even. Suppose  $n(\pi) \leq \frac{p+q}{2} - 1$ , which implies that  $\zeta = \xi = 1$  by Lemma 24. Let  $n, k$  be any integers such that  $n \geq n(\pi)$ ,  $k \geq 1$ , and  $n \notin [\frac{p+q}{2}, \frac{p+q}{2} + k - 1]$ . (If  $n(\pi) \leq n \leq \frac{p+q}{2} - 1$ , we may take any  $k \geq 1$ .) Then*

$$\theta_{p+k, q+k}(\theta_n(\pi)) = \pi_1(\lambda_d, 1, \Psi, \mu, \nu, (\varepsilon|(1, 1, \dots, 1)), \\ (\kappa|(n - \frac{p+q}{2}, n-1 - \frac{p+q}{2}, \dots, n-k+1 - \frac{p+q}{2})))$$

with a possible modification: if the resulting parameters contain entries  $\varepsilon_i \neq \varepsilon_j$  and  $\kappa_i = \pm \kappa_j$ , delete  $\varepsilon_i, \varepsilon_j, \kappa_i, \kappa_j$  from  $(\varepsilon, \kappa)$ , and add entries  $(0, 2\kappa_i)$  into  $(\mu, \nu)$ .

*Proof.* Since  $n(\pi) \leq \frac{p+q}{2} - 1$ , the parameters of  $\pi$  satisfy (ii) of Lemma 24 and  $\mathcal{A}(\pi) \subseteq \mathcal{D}(\pi)$ . Take any  $\sigma \in \mathcal{A}(\pi)$ , we get  $\sigma_{1,1} \in \mathcal{D}(\theta_{p+1, q+1}(\theta_n(\pi)))$ . By Proposition 23,  $\sigma_{1,1}$  is also a lowest  $K$ -type of  $\pi_{1,1}$ . By Proposition 21,  $\theta_{p+1, q+1}(\theta_n(\pi)) = \pi_1(\lambda_d, 1, \Psi, \mu, \nu, (\varepsilon|(1)), (\kappa|(n - \frac{p+q}{2})))$  with a possible modification of parameters. Repeat this process for  $k$  times.  $\square$

#### 4. REDUCING CASES WHEN $p+q=4$

The following parts of this paper is to explicitly compute the theta liftings

$$\theta_n : \mathcal{R}(O(p, q)) \rightarrow \mathcal{R}(Sp(2n, \mathbb{R})) \cup \{0\},$$

for all  $n \geq 1$  when  $p+q=4$ , in terms of Langlands parameters. In this section we reduce the cases that need computation.

**4.1. Reducing  $(p, q)$ .** Since  $O(p, q) = O(q, p)$ , there is a bijection  $\varphi : \mathcal{R}(O(p, q)) \xrightarrow{\sim} \mathcal{R}(O(q, p))$ . Under our parametrization, then  $\varphi$  is indeed the following operations on  $(\lambda_d, \Psi)$  (preserving other parameters):

- interchange the two parts of  $\lambda_d = \{*, *\}$ ,
- interchange the roles of  $\{e_i\}$  and  $\{f_j\}$  for  $\Psi$ .

**Lemma 26** ([Pau05, Lemma 20]). *Let  $n, p$ , and  $q$  be nonnegative integers with  $p+q$  even. Let  $\pi' \in \mathcal{R}(Sp(2n, \mathbb{R}))$  with  $\theta_{p,q}(\pi') \neq 0$  and let  $\pi'^*$  denote its contragredient. Then*

$$\varphi(\theta_{p,q}(\pi')) = \theta_{q,p}(\pi'^*).$$

To transfer between the Langlands parameters of  $\pi'$  and  $\pi'^*$ , we only need to

- replace  $\lambda_d = (c_1, c_2, \dots, c_v)$  by  $(-c_v, -c_{v-1}, \dots, -c_1)$ ,
- replace  $e_i$  by  $-e_{v+1-i}$  for  $\Psi$ .

Therefore, to compute  $\theta_n$  for  $O(p, q)$  when  $p+q=4$ , it suffices to compute when  $(p, q) = (4, 0), (3, 1), (2, 2)$ .



**4.2. First occurrence.** For  $\pi \in \mathcal{R}(O(p, q))$  with  $p + q$  even, its *first occurrence index* is

$$n(\pi) = \min\{k \geq 0 \mid \theta_k(\pi) \neq 0\}.$$

Taking the oscillator representation of  $\widetilde{Sp}(0, \mathbb{R}) \cong \mu_2 = \{\pm 1\}$  to be the nontrivial character, we get the local theta correspondence for  $(O(p, q), Sp(0, \mathbb{R}))$  as  $\mathbb{1} \leftrightarrow \mathbb{1}$ , where  $\mathbb{1}$  means the trivial representation. In this sense, it is assumed that  $n(\mathbb{1}) = 0$ , and  $n(\pi) \geq 1$  for any nontrivial  $\pi$ .

A reductive dual pair  $(G, G')$  of type I is said to be in the stable range with  $G$  the smaller member if the defining module of  $G'$  has an isotropic subspace of the same dimension as that of the defining module of  $G$ . For  $(G, G') = (O(p, q), Sp(2n, \mathbb{R}))$  with  $p + q$  even, it is in the stable range with  $G$  the smaller member if  $n \geq p + q$ . The nonvanishing of theta liftings in the stable range (cf. [Li89, PP08]) states that:

**Proposition 27.**  $n(\pi) \leq p + q$  for all  $\pi \in \mathcal{R}(O(p, q))$  with  $p + q$  even.

Let  $\det$  denotes the determinant character of  $O(p, q)$ . The only  $O(p) \times O(q)$ -type occurring in  $\det$  is  $(0, \dots, 0; -1) \otimes (0, \dots, 0; -1)$ , which does not occur in the space of joint harmonics when  $n < p + q$  by Proposition 4. Thus by Lemma 5,  $n(\det) \geq p + q$ , and hence  $n(\det) = p + q$ .

Recently B. Sun and C.-B. Zhu [SZ14] proved some interesting “conservation relations” conjectured by Kudla-Rallis ([KR05]) about the first occurrence for local theta correspondence. Especially, the following strong relation for  $(O(p, q), Sp(2n, \mathbb{R}))$  holds.

**Lemma 28** ([SZ14]). For  $\pi \in \mathcal{R}(O(p, q))$  with  $p + q$  even,

$$n(\pi) + n(\pi \otimes \det) = p + q.$$

**Proposition 29.** Let  $\pi \in \mathcal{R}(O(p, q))$  with  $p + q$  even. If  $\pi \neq \det$ , then  $n(\pi) \leq p + q - 1$ .

*Proof.* We see  $\pi \otimes \det \neq \mathbb{1}$ , so  $n(\pi \otimes \det) \geq 1$  and  $n(\pi) = p + q - n(\pi \otimes \det) \leq p + q - 1$ .  $\square$

Therefore,  $\det$  is the only element in  $\mathcal{R}(O(p, q))$  with first occurrence index  $p + q$ .

Fix  $(p, q)$  with  $p + q$  even and  $\pi \in \mathcal{R}(O(p, q))$ . By the explicit induction principle on  $n$  (Theorem 19), to explicitly compute  $\theta_n(\pi)$  for all  $n$ , it suffices to compute  $\theta_n(\pi)$  for  $n(\pi) \leq n \leq \max\{n(\pi), \frac{p+q}{2}\}$ .

When  $p + q = 4$ ,  $\theta_1$  and  $\theta_2$  can be explicitly read off from [Pau05] as (almost) equal rank cases, and will be written down in Appendix B. So we only need to compute  $\theta_4(\det)$  and  $\theta_3(\pi)$  for all  $\pi \in \mathcal{R}(O(p, q))$  with  $n(\pi) = 3$ .

## 5. THETA 4-LIFTS OF $\det$ WHEN $p + q = 4$

To distinguish different  $(p, q)$ , let  $\det_{p,q}$  denote the determinant character of  $O(p, q)$ . In this section, we will compute  $\theta_4(\det_{p,q})$  explicitly when  $(p, q) = (4, 0), (3, 1), (2, 2)$ . The strategy is to determine the desired theta lifts by its infinitesimal character and lowest  $K$ -types.

First notice that the infinitesimal character of  $\det_{p,q}$  is  $(0, 1)$ . By Proposition 12, the infinitesimal character of  $\theta_4(\det_{p,q})$  is  $(0, 1, 1, 2)$ . Also notice that  $\mathcal{A}(\theta_4(\det_{p,q})) \subseteq \mathcal{D}(\theta_4(\det_{p,q}))$  by Lemma 17. The only  $K$ -type of  $\det_{2,2}$  is  $(0; -1) \otimes (0; -1)$ . By Proposition 4 and Lemma 5,

$$\mathcal{A}(\theta_4(\det_{2,2})) = \mathcal{D}(\theta_4(\det_{2,2})) = \{(1, 1, -1, -1)\}.$$

Similarly, the only  $K$ -type of  $\det_{3,1}$  is  $(0; -1) \otimes (0; -1)$  and

$$\mathcal{A}(\theta_4(\det_{3,1})) = \mathcal{D}(\theta_4(\det_{3,1})) = \{(2, 2, 2, 0)\};$$

the only  $K$ -type of  $\det_{4,0}$  is  $(0, 0; -1) \otimes (; )$  and

$$\mathcal{A}(\theta_4(\det_{4,0})) = \mathcal{D}(\theta_4(\det_{4,0})) = \{(3, 3, 3, 3)\}.$$

**5.1. Determinant for  $O(2, 2)$ .** Let us compute  $\theta_{2p}(\det_{p,p}) \in \mathcal{R}(Sp(4p, \mathbb{R}))$  for any  $p \geq 1$ . The infinitesimal character of  $\det_{p,p}$  is  $(0, 1, 2, \dots, p-1)$ . By Proposition 12, the infinitesimal character of  $\theta_{2p}(\det_{p,p})$  is

$$(0, 1, 1, 2, 2, \dots, p-1, p-1, p)$$

The only  $O(p) \times O(p)$ -type of  $\det_{p,p}$  is the  $(0, \dots, 0; -1) \otimes (0, \dots, 0; -1)$ , so

$$\mathcal{A}(\theta_{2p}(\det_{p,p})) = \{(\underbrace{1, \dots, 1}_p, \underbrace{-1, \dots, -1}_p)\}.$$

**Proposition 30.** *For  $1 \leq p \in \mathbb{Z}$ , there exists a unique  $\pi' \in \mathcal{R}(Sp(4p, \mathbb{R}))$  with infinitesimal character  $(0, 1, 1, 2, 2, \dots, p-1, p-1, p)$  and  $\mathcal{A}(\pi') = \{(\underbrace{1, \dots, 1}_p, \underbrace{-1, \dots, -1}_p)\}$ , which is*

$$\theta_{2p}(\det_{p,p}) = \pi(0, \emptyset, (1, 1, \dots, 1), (1, 3, 5, \dots, 2p-1), 0, 0).$$

*Proof.* Let  $\pi' = \pi(\lambda_d, \Psi, \mu, \nu, \varepsilon, \kappa)$ . In the algorithm to calculate  $\mathcal{A}(\pi')$  (Proposition 44),

$$\begin{aligned} & \lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) \\ &= (\underbrace{\beta_1, \dots, \beta_1}_{u_1}, \dots, \underbrace{\beta_m, \dots, \beta_m}_{u_m}, \underbrace{u-r, \dots, u-r}_w, \underbrace{\gamma_m, \dots, \gamma_m}_{r_1}, \dots, \underbrace{\gamma_1, \dots, \gamma_1}_{r_m}) \end{aligned}$$

with  $\beta_1 \geq \dots \geq \beta_m \geq u-r+1 > u-r-1 \geq \gamma_m \geq \dots \geq \gamma_1$ . Here  $u = \sum_{i=1}^m u_i$ ,  $r = \sum_{i=1}^m r_i$ , and  $w$  are the number of positive, negative, and zero entries in  $\lambda_a$  respectively. To get the lowest  $K$ -type  $\sigma = (\underbrace{1, \dots, 1}_p, \underbrace{-1, \dots, -1}_p)$ , every  $\beta_i$  with  $u_i > 0$  or  $\gamma_i$  with  $r_i > 0$  should lie in

$$\{0, \pm 1, \pm \frac{3}{2}\}.$$

First we show  $u = r$ . If  $u > r$ , then all  $\beta_i \geq u-r+1 \geq 2$ , and thus  $0 = u > r$ , making a contradiction. If  $u < r$  then all  $\gamma_i \leq u-r-1 \leq -2$ , and thus  $0 = r > u$  making a contradiction.

Since  $u-r=0$ , the lowest  $K$ -type  $\sigma$  is of the form

$$(\underbrace{*, \dots, *}_u, \underbrace{1, \dots, 1}_h, \underbrace{0, \dots, 0}_{w-h}, \underbrace{*, \dots, *}_u) \text{ or } (\underbrace{*, \dots, *}_u, \underbrace{0, \dots, 0}_{w-h}, \underbrace{-1, \dots, -1}_h, \underbrace{*, \dots, *}_u).$$

Since  $\sigma$  have  $p$  entries 1 and  $p$  entries  $-1$ , we have  $w = h = 0$ . Therefore,  $(\varepsilon, \kappa)$  does not occur, and  $\lambda_d$  and  $\mu$  contains no zero entry.

Let  $\alpha_1$  be the maximal absolute value of the entries of  $\lambda_a$ . Clearly  $\lambda_a \neq (0, \dots, 0)$ . So  $\frac{1}{2} \leq \alpha_1 \in \mathbb{Z} + \frac{1}{2}$ . Since  $u_1 > 0$  or  $r_1 > 0$ , we have  $\beta_1 = \alpha_1 + \frac{u_1-r_1}{2} + \frac{1}{2} \leq \frac{3}{2}$  or  $\gamma_1 = -\alpha_1 + \frac{u_1-r_1}{2} - \frac{1}{2} \geq -\frac{3}{2}$ . As  $|u_1 - r_1| \leq 1$ , we get  $\alpha_1 \leq \frac{3}{2}$ . But when  $\alpha_1 = \frac{3}{2}$ , we must have  $u_1 = r_1$ , and then  $\alpha_1 \leq 1$ , which makes a contradiction. So  $\alpha_1 \in \{\frac{1}{2}, 1\}$ .

Therefore,  $\mu_i \in \{1, 2\}$ , and the entries of  $\lambda_d$  lie in  $\{0, \pm 1\}$ . Let

$$\lambda_d = (\underbrace{a_1, \dots, a_1}_{k_1}, \dots, \underbrace{a_b, \dots, a_b}_{k_b}, \underbrace{0, \dots, 0}_z, \underbrace{-a_b, \dots, -a_b}_{l_1}, \dots, \underbrace{-a_1, \dots, -a_1}_{l_b})$$

with  $a_1 > a_2 > \dots > a_b > 0$ , and  $|k_i - l_i| \leq 1$  for all  $i$ . Let  $k = \sum_{i=1}^b k_i$ ,  $l = \sum_{i=1}^b l_i$ . If  $a_1$  occurs, then  $a_1 = 1$ . So  $b = 1$  or  $0$ . Since  $k-l = u-r = 0$ , we have  $k_1 = l_1 = k = l$ . Therefore,  $u_1 = r_1 \geq 0$ .

We show that  $\alpha_1 \neq 1$ . Otherwise, suppose  $\alpha_1 = 1$ . Then  $\beta_1 = \alpha_1 + \frac{u_1-r_1}{2} + \frac{1}{2} = \frac{3}{2}$  and  $\gamma_1 = -\frac{3}{2}$ , both of which are entries of  $\lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k})$ . In spite of the choice of  $\pm \frac{1}{2}$  adding into  $\beta_1$  and  $\gamma_1$ , we get 2 or  $-2$  as an entry of the lowest  $K$ -type. This is not true for  $\sigma$ .

So  $\alpha_1 = \frac{1}{2}$ . Therefore,  $\lambda_d$  do not occur and  $0 < \mu_i \leq 1$  for all  $i$ .

In all,  $\pi' = \pi(0, \emptyset, \mu, \nu, 0, 0)$  with  $\mu = (1, 1, \dots, 1)$ . The infinitesimal character of  $\pi'$  is

$$(0, 1, 1, \dots, p-1, p-1, p) \sim \left(\frac{1+\nu_1}{2}, \frac{1-\nu_1}{2}, \frac{1+\nu_2}{2}, \frac{1-\nu_2}{2}\right).$$

Here  $\sim$  denotes the equivalence up to permutations and sign-changes of coordinates. So  $\nu = (1, 3, 5, \dots, 2p-1)$  is the only choice (up to equivalence). The uniqueness of  $\pi'$  ensures that it is also  $\theta_{2p}(\det_{p,p})$ .  $\square$

## 5.2. Determinant for $O(3, 1)$ .

**Proposition 31.** *There exists a unique  $\pi' \in \mathcal{R}(Sp(8, \mathbb{R}))$  with infinitesimal character  $(0, 1, 1, 2)$  and  $\mathcal{A}(\pi') = \{(2, 2, 2, 0)\}$ , which is*

$$\theta_4(\det_{3,1}) = \pi((1, 0), \Psi, (1), (3), 0, 0)$$

with  $\Psi = \{e_1 \pm e_2, 2e_1, 2e_2\}$ .

*Proof.* Let  $\pi' = \pi(\lambda, \Psi, \mu, \nu, \varepsilon, \kappa)$ . Review the algorithm to calculate  $\mathcal{A}(\pi')$  (Proposition 44).

First we show  $u - r = 1$ . If  $u - r \geq 2$ , then all  $\beta_i \geq u - r + 1 \geq 3$  which cannot occur, and thus  $0 = u > r$  making a contradiction. If  $u - r \leq 0$ , then all  $\gamma_i \leq u - r - 1 \leq -1$  which cannot occur, and thus  $0 = r \geq u$ , and  $w = 4$ , and  $\sigma$  has only entries in  $\{0, \pm 1\}$  making a contradiction.

As  $u - r = 1$  and  $u + w + r = 4$ , we see  $(u, w, r) = (2, 1, 1)$  or  $(1, 3, 0)$ . Notice that  $\sigma = (2, 2, 2, 0)$  is of the form

$$\underbrace{(*, \dots, *)}_u, \underbrace{2, \dots, 2}_h, \underbrace{1, \dots, 1}_{w-h}, \underbrace{*, \dots, *)}_r \text{ or } \underbrace{(*, \dots, *)}_u, \underbrace{1, \dots, 1}_{w-h}, \underbrace{0, \dots, 0}_h, \underbrace{*, \dots, *)}_r,$$

so  $h = w$ . The second form cannot happen, otherwise  $w + r = 1$  (the number of zero entries in  $\sigma$ ). So  $r = 1$ , and  $(u, w, r, h) = (2, 1, 1, 1)$ . If  $z = 0$ , both above two forms should occur, making lowest  $K$ -types not unique. So  $z > 0$ . As  $z \leq w = 1$ , we get  $z = w = 1$ , which implies that:  $\lambda_d$  contains exactly one zero entry;  $(\varepsilon, \kappa)$  do not occur; and  $\mu$  contains no zero entry. Hence  $(v, s, t) = (2, 1, 0)$  or  $(4, 0, 0)$ .

We show  $(v, s, t) \neq (4, 0, 0)$ . Otherwise,  $\lambda_d \sim (0, 1, 1, 2)$ , with  $k = u = 2$  and  $l = r = 1$ . Then  $\lambda_d = (2, 1, 0, -1)$  and  $\lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) = (3, \frac{5}{2}, 1, -\frac{1}{2})$ , which cannot give the lowest  $K$ -type  $\sigma$ .

Thus  $(v, s, t) = (2, 1, 0)$ . As  $k - l = u - r = 1$  and  $k + z + l = v$ , we get  $(k, z, l) = (1, 1, 0)$ . So  $\lambda_d = (a_1, 0)$  for  $1 \leq a_1 \in \mathbb{Z}$ . Now  $(a_1, \frac{\mu_1+\nu_1}{2}, \frac{-\mu_1+\nu_1}{2}) \sim (1, 1, 2)$ . Notice that  $\mu_1 > 0$ , and that  $\mu_1$  should be odd if  $\nu_1 = 0$ . So  $(\frac{\mu_1+\nu_1}{2}, \frac{-\mu_1+\nu_1}{2}) \not\sim (1, 1)$ . Thus  $(\frac{\mu_1+\nu_1}{2}, \frac{-\mu_1+\nu_1}{2}) \sim (1, 2)$  and  $\lambda_d = (1, 0)$ . Then  $(a_1, \mu_1, \nu_1) = (1, 1, \pm 3)$  or  $(1, 3, \pm 1)$ , and  $\lambda_a = (1, \frac{1}{2}, 0, -\frac{1}{2})$  or  $(\frac{3}{2}, 1, 0, -\frac{3}{2})$ , with the corresponding  $\lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) = (2, 2, 1, 0)$  or  $(2, 2, 1, -2)$  respectively. But the latter one cannot give the lowest  $K$ -type  $\sigma$ . So  $\lambda_a = (1, \frac{1}{2}, 0, -\frac{1}{2})$ ,  $\mu = (1)$ , and  $\nu \sim (3)$ . In the algorithm, to add  $h$  entries 1 on the  $w$  entries, we shall have  $2e_2 \in \Psi$ .

In all,  $\pi' = \pi((1, 0), \Psi, (1), (3), 0, 0)$  with  $\Psi = \{e_1 \pm e_2, 2e_1, 2e_2\}$ . The uniqueness ensures that it is also  $\theta_4(\det_{3,1})$ .  $\square$

### 5.3. Determinant for $O(4, 0)$ .

**Proposition 32.** *There exists a unique  $\pi' \in \mathcal{R}(Sp(8, \mathbb{R}))$  with infinitesimal character  $(0, 1, 1, 2)$  and  $\mathcal{A}(\pi') = \{(3, 3, 3, 3)\}$ , which is*

$$\theta_4(\det_{4,0}) = \pi((2, 1, 0), \Psi, 0, 0, (-1), (1)),$$

with  $\Psi = \{e_1 \pm e_2, e_2 \pm e_3, e_1 \pm e_3, 2e_1, 2e_2, 2e_3\}$ .

*Proof.* Let  $\pi' = \pi(\lambda, \Psi, \mu, \nu, \varepsilon, \kappa)$ . Review the algorithm to calculate  $\mathcal{A}(\pi')$  (Proposition 44).

First we show  $u - r = 2$ . If  $u - r \geq 3$ , then all  $\beta_i \geq u - r + 1 \geq 4$  which cannot occur, and thus  $0 = u > r$ , which makes a contradiction. If  $u - r \leq 1$ , then all  $\gamma_i \leq u - r - 1 \leq 0$ , which cannot occur, and thus  $0 = r \geq u - 1$  and  $w = 4 - u \geq 3$ . The  $w$  entries  $(u - r, \dots, u - r)$  occur, so  $u - r \in \{2, 3, 4\}$ , which makes a contradiction.

As all  $\gamma_i \leq u - r - 1 = 1$  cannot occur,  $r = 0$ . So  $u = 2$ ,  $w = 2$ . Clearly  $h = w = 2$ . If  $z = 0$ , the lowest  $K$ -types will not be unique. So  $z \geq 1$ . And  $z - \lfloor \frac{z+1}{2} \rfloor \leq w - h = 0$ , so  $z = 1$ . Since  $w = h = 2$ , we get all  $\varepsilon_i = -1$ .

As  $k - l = u - r = 2$  and  $z = 1$ , the odd integer  $v = k + z + l \geq 3$ . So  $(v, s, t) = (3, 0, 1)$ . Since  $2 = v - z = k + l \geq k - l = 2$ , we have  $k = 2$  and  $l = 0$ . Notice that  $(\lambda_d \mid \kappa) \sim (0, 1, 1, 2)$ , and  $\lambda_d$  contains an entry 0. We must have  $\lambda_d = (2, 1, 0)$ , since  $(1, 1, 0)$  do not satisfy  $|k_1 - l_1| \leq 1$ . Thus  $\kappa \sim (1)$ . In the algorithm, to add  $h$  entries 1 on the  $w$  entries, we shall have  $2e_3 \in \Psi$ .

In all, we get  $\pi' = \pi((2, 1, 0), \Psi, 0, 0, (-1), (1))$  with  $\Psi = \{e_1 \pm e_2, e_2 \pm e_3, e_1 \pm e_3, 2e_1, 2e_2, 2e_3\}$ . The uniqueness ensures that it is also  $\theta_4(\det_{4,0})$ .  $\square$

The results of this section can be collected in the following table.

$(p, q)$	$\det = \det_{p,q}$	inf. char. of $\theta_4(\det)$	$\mathcal{A}(\theta_4(\det))$	$\theta_4(\det)$	$\Psi$
$(2, 2)$	$\pi_{-1}(0, 1, \emptyset, 0, 0, (1, 1), (0, 1))$	$(0, 1, 1, 2)$	$\{(1, 1, -1, -1)\}$	$\pi(0, \emptyset, (1, 1), (1, 3), 0, 0)$	$\emptyset$
$(3, 1)$	$\pi_1((0;), -1, \emptyset, 0, 0, (1), (1))$		$\{(2, 2, 2, 0)\}$	$\pi((1, 0), \Psi, (1), (3), 0, 0)$	$\{e_1 \pm e_2, 2e_1, 2e_2\}$
$(4, 0)$	$\pi_1((1, 0;), -1, \{e_1 \pm e_2\}, 0, 0, 0, 0)$		$\{(3, 3, 3, 3)\}$	$\pi((2, 1, 0), \Psi, 0, 0, (-1), (1))$	$\{e_1 \pm e_2, e_2 \pm e_3, e_1 \pm e_3, 2e_1, 2e_2, 2e_3\}$

### 6. THETA 3-LIFTS WHEN $p + q = 4$ AND $n(\pi) = 3$

In this section  $\theta_3(\pi)$  is determined explicitly for any  $\pi \in \mathcal{R}(O(p, q))$  with  $n(\pi) = 3$  when  $(p, q) = (4, 0), (3, 1), (2, 2)$ .

**Lemma 33** ([Pau05, Corollary 24]). *Let  $\pi = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa) \in \mathcal{R}(O(p, q))$  with  $p + q$  even. Then  $n(\pi) \leq \frac{p+q}{2}$  if and only if either  $\xi = \zeta = 1$ , or  $\lambda_d$  contains no zero entry and  $(\varepsilon_i, \kappa_i) = (-1, 0)$  for some  $i$ .*

**Proposition 34.** *Let  $\pi = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa) \in \mathcal{R}(O(p, q))$  with  $p + q = 4$ . Then  $n(\pi) = 3$  if and only if  $\pi \neq \det$  and either  $\xi = -1$ , or  $\zeta = -1$  and  $(\varepsilon_i, \kappa_i) = (1, 0)$  for some  $i$ .*

*Proof.* The above lemma states that  $n(\pi) \geq 3 \Leftrightarrow (\xi, \zeta) \neq (1, 1)$ , and either  $\lambda_d$  contains a zero entry or all  $(\varepsilon_i, \kappa_i) \neq (-1, 0)$ . Notice that  $\xi = -1$  only if  $\lambda_d$  contains a zero entry. Also notice that  $\zeta = -1$  only if  $\lambda_d$  contains no zero entry and  $\kappa$  contains a zero entry.  $\square$

By this proposition, when  $p + q = 4$ , we can list the explicit Langlands parameters for all  $\pi \in \mathcal{R}(O(p, q))$  with  $n(\pi) = 3$ .

**6.1. For  $O(4, 0)$ .** Each  $\pi \in \mathcal{R}(O(4, 0))$  is parametrized as  $\pi = \pi_1((m, l; ), \xi, \Psi, 0, 0, 0, 0)$  with integers  $m > l \geq 0$  and  $\xi \in \{\pm 1\}$ . Now  $\Delta = \{\pm e_1 \pm e_2\}$ ,  $\Delta_c^+ = \{e_1 \pm e_2\} = \Psi$ . Here  $\Psi$  is fixed.

Now  $O(4, 0) = O(4) \times O(0)$  is compact, so  $\pi$  itself is a  $K$ -type for  $O(4, 0)$ . By Proposition 45 and 46 to compute lowest  $K$ -types, this  $K$ -type is  $(m - 1, l; \xi) \otimes (; )$ . So

$$\det_{4,0} = \pi_1((1, 0; ), -1, \{e_1 \pm e_2\}, 0, 0, 0, 0).$$

By Proposition 34, all  $\pi \in \mathcal{R}(O(4, 0))$  with  $n(\pi) = 3$  are

$$\pi = \pi_1((m, 0; ), -1, \{e_1 \pm e_2\}, 0, 0, 0, 0) \quad \text{with } 2 \leq m \in \mathbb{Z}.$$

For such  $\pi$ , by Lemma 17,

$$\begin{aligned} \mathcal{A}(\theta_3(\pi)) &= \mathcal{D}(\theta_3(\pi)) = \{\phi_3((m - 1, 0; -1) \otimes (; ))\}. \\ &= \{(m + 1, 3, 3)\}. \end{aligned}$$

The infinitesimal character of  $\pi$  is  $(m, 0)$ . By Proposition 12, the infinitesimal character of  $\theta_3(\pi)$  is  $(m, 0, 1)$ .

**Proposition 35.** *For  $2 \leq m \in \mathbb{Z}$ , there is a unique  $\pi' \in \mathcal{R}(Sp(6, \mathbb{R}))$  with infinitesimal character  $(m, 0, 1)$  and  $\mathcal{A}(\pi') = \{(m + 1, 3, 3)\}$ . It is*

$$\theta_3(\pi_1((m, 0; ), -1, \{e_1 \pm e_2\}, 0, 0, 0, 0)) = \pi((m, 1, 0), \Psi, 0, 0, 0, 0),$$

with  $\Psi = \{e_1 \pm e_2, e_2 \pm e_3, e_1 \pm e_3, 2e_1, 2e_2, 2e_3\}$ .

*Proof.* Go through Appendix C which lists  $\mathcal{A}(\pi')$  for all  $\pi' \in \mathcal{R}(Sp(6, \mathbb{R}))$  with infinitesimal character  $(\beta, 0, 1)$ , where  $\beta \in \mathbb{C}$ .  $\square$

**6.2. For  $O(2, 2)$ .** Let  $\pi = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa) \in \mathcal{R}(O(2, 2))$  with  $n(\pi) = 3$ . By Proposition 34, all these  $\pi$  are listed in the following table.

$\zeta$	$\lambda_d$	$\xi$	$\Psi$	$\mu$	$\nu$	$\varepsilon$	$\kappa$	with		
1	$(n; 0)$	-1	$\{e_1 \pm f_1\}$	0	0	0	0	$0 \leq n \in \mathbb{Z}$		
	$(0; n)$		$\{\pm e_1 + f_1\}$							
-1	0	1	$\emptyset$			$(1, -1)$	$(0, \beta)$	$\beta \in \mathbb{C} \backslash \{0\}$ by (F-2)		
						$(1, 1)$	$(0, \beta)$	$\beta \in \mathbb{C} \backslash \{\pm 1\}$		

*Remark.*  $\det_{2,2} = \pi_{-1}(0, 1, \emptyset, 0, 0, (1, 1), (0, 1))$ .

**Proposition 36.** *If  $\pi = \pi_1((n; 0), -1, \{e_1 \pm f_1\}, 0, 0, 0, 0) \in \mathcal{R}(O(2, 2))$  with  $0 \leq n \in \mathbb{Z}$ , then  $\mathcal{A}(\theta_3(\pi)) = \{(n + 1, -1, -1)\}$ .*

*Proof.* By Proposition 45 and 46,  $\mathcal{A}(\pi) = \{(n + 1; 1) \otimes (0; -1)\}$ . The lowest  $K$ -type of  $\pi$  has degree  $n + 3$ , so  $\deg(\pi) \leq n + 3$ . We claim that  $\deg(\pi) = n + 3$ . Otherwise, by Lemma 6 on the parity of degrees, there is a  $K$ -type  $(m; \epsilon) \otimes (l; \eta)$  of  $\pi$  with degree  $\leq n + 1$ . So  $m + l \leq n + 1$ . But  $(m; \epsilon) \otimes (l; \eta)$  is not a lowest  $K$ -type of  $\pi$ , so  $m^2 + l^2 > (n + 1)^2 \geq (m + l)^2$ , which makes a contradiction.

All  $K$ -types for  $O(2, 2)$  occurring in the space of joint harmonics  $\mathcal{H}$  with degree  $n + 3$  are:  $(n + 1; 1) \otimes (0; -1)$ ,  $(0; -1) \otimes (n + 1; 1)$ , and  $(m; 1) \otimes (l; 1)$  with  $m + l = n + 3$ . As  $(0; -1) \otimes (n + 1; 1)$  has the same norm as that of the lowest  $K$ -type of  $\pi$ , it cannot occur in  $\pi$ . Notice that  $(m; 1) \otimes (l; 1)$

occurs in  $\pi$  only if  $m^2 + l^2 > (n+1)^2$ . Therefore, all  $U(3)$ -types in  $\mathcal{D}(\theta_3(\pi))$  other than  $(n+1, -1, -1)$  are of the form  $(m, 0, -l)$  with  $m+l = n+3$  and  $m^2 + l^2 > (n+1)^2$ . Then

$$\begin{aligned} \|(m, 0, -l)\| &= m^2 + l^2 + 4(m+l) + 8 \\ &> (n+1)^2 + 4(n+3) + 8 \\ &> \|(n+1, -1, -1)\|. \end{aligned}$$

By Lemma 17, the set  $\mathcal{A}(\theta_3(\pi))$  consists of  $U(3)$ -types with minimal norm in  $\mathcal{D}(\theta_3(\pi))$ , so  $\mathcal{A}(\theta_3(\pi)) = \{(n+1, -1, -1)\}$ .  $\square$

*Remark.* Similarly, if  $\pi = \pi_1((0; n), -1, \{\pm e_1 + f_1\}, 0, 0, 0, 0)$  with  $0 \leq n \in \mathbb{Z}$ , then  $\mathcal{A}(\theta_3(\pi)) = \{(1, 1, -n-1)\}$ .

**Proposition 37.** *If  $\pi = \pi_{-1}(0, 1, \emptyset, 0, 0, (1, -1), (0, \beta))$  with  $\beta \neq 0$ , then*

$$\mathcal{A}(\theta_3(\pi)) = \{(1, -1, -1), (1, 1, -1)\}.$$

*Proof.* By Proposition 45 and 46, we get  $\mathcal{A}(\pi) = \{(1; 1) \otimes (0; -1), (0; -1) \otimes (1; 1)\}$ . The lowest  $K$ -types of  $\pi$  have degree 3, so  $\deg(\pi) = 1$  or 3 by Lemma 6 on the parity of degrees. All  $K$ -types for  $O(2, 2)$  with degree = 1 are  $(1; 1) \otimes (0; 1)$  and  $(0; 1) \otimes (1; 1)$ . They all have norms = the norm of lowest  $K$ -types of  $\pi$ , thus cannot occur in  $\pi$ . So  $\deg(\pi) = 3$  and  $\mathcal{A}(\pi) \subset \mathcal{D}(\pi)$ . Then

$$\phi_3(\mathcal{A}(\pi)) = \{(1, -1, -1), (1, 1, -1)\} \subset \phi_3(\mathcal{D}(\pi)) = \mathcal{D}(\theta_3(\pi)).$$

All  $K$ -types for  $Sp(6, \mathbb{R})$  occurring in the space of joint harmonics  $\mathcal{H}$  with degree 3 are:  $(1, -1, -1)$ ,  $(1, 1, -1)$ ,  $(2, 0, -1)$ ,  $(1, 0, -2)$ ,  $(3, 0, 0)$ , and  $(0, 0, -3)$ , among which the first two take the minimal norm and belong to  $\mathcal{D}(\theta_3(\pi))$ , so  $\mathcal{A}(\theta_3(\pi)) = \{(1, -1, -1), (1, 1, -1)\}$  by Lemma 17.  $\square$

**Proposition 38.** *If  $\pi = \pi_{-1}(0, 1, \emptyset, 0, 0, (1, 1), (0, \beta))$  with  $\beta \neq \pm 1$ , then  $\mathcal{A}(\theta_3(\pi)) = \{(1, 0, -1)\}$ .*

*Proof.* By Proposition 45 and 46, we get  $\mathcal{A}(\pi) = \{(0; -1) \otimes (0; -1)\}$ . The lowest  $K$ -type of  $\pi$  has degree 4, but does not occur in the space of joint harmonics  $\mathcal{H}$  for  $(O(2, 2), Sp(6, \mathbb{R}))$ . Thus  $\deg(\pi) < 4$ . So  $\deg(\pi) = 0$  or 2 by Lemma 6 on the parity of degrees. The only  $K$ -type for  $O(2, 2)$  with degree 0 is  $(0; 1) \otimes (0; 1)$ . It has norm = the norm of the lowest  $K$ -type of  $\pi$ , thus cannot occur in  $\pi$ . So  $\deg(\pi) = 2$ .

All  $K$ -types for  $O(2, 2)$  with degree 2 are:  $(0; -1) \otimes (0; 1)$ ,  $(0; 1) \otimes (0; -1)$ ,  $(1; 1) \otimes (1; 1)$ ,  $(2; 1) \otimes (0; 1)$  and  $(0; 1) \otimes (2; 1)$ . The first two have norm = the norm of the lowest  $K$ -type of  $\pi$ , and thus cannot occur in  $\pi$ . Therefore,

$$\begin{aligned} \mathcal{D}(\pi) &\subset \{(1; 1) \otimes (1; 1), (2; 1) \otimes (0; 1), (0; 1) \otimes (2; 1)\}, \\ \mathcal{D}(\theta_3(\pi)) &= \phi_3(\mathcal{D}(\pi)) \subset \{(1, 0, -1), (2, 0, 0), (0, 0, -2)\}. \end{aligned}$$

And the norms  $\|(1, 0, -1)\| < \|(2, 0, 0)\| = \|(0, 0, -2)\|$ .

By Lemma 26,  $\theta_3(\pi)^* = \theta_3(\varphi(\pi)) = \theta_3(\pi)$ , which implies that  $\theta_3(\pi)$  has parameter  $\lambda_d$  of the form  $(c, 0, -c)$  and  $\Psi$  unchanged after replacing  $e_i$  by  $-e_{v+1-i}$ . By the algorithm in Proposition 44, the set  $\mathcal{A}(\theta_3(\pi))$  remains the same after replacing each  $U(3)$ -type  $(m, n, l)$  by  $(-l, -n, -m)$ .

By Lemma 17, the set  $\mathcal{A}(\theta_3(\pi))$  consists of  $K$ -types in  $\mathcal{D}(\theta_3(\pi))$  with minimal norm, so  $\mathcal{A}(\theta_3(\pi)) = \{(1, 0, -1)\}$  or  $\{(2, 0, 0), (0, 0, -2)\}$ . The infinitesimal character of  $\pi$  is  $(\beta, 0)$ , thus that of  $\theta_3(\pi)$  is  $(\beta, 0, 1)$ . According to the Appendix C which lists  $\mathcal{A}(\pi')$  for all  $\pi' \in \mathcal{R}(Sp(6, \mathbb{R}))$  with infinitesimal character  $(\beta, 0, 1)$ , we see  $\mathcal{A}(\pi') \neq \{(2, 0, 0), (0, 0, -2)\}$ .  $\square$

**Theorem 39.** *Let  $\pi \in \mathcal{R}(O(2, 2))$  with  $n(\pi) = 3$ .*

(1) *If  $\pi \neq \pi_{-1}(0, 1, \emptyset, 0, 0, (1, 1), (0, 2))$ , then the infinitesimal character of  $\theta_3(\pi)$  and the set  $\mathcal{A}(\theta_3(\pi))$  determine a unique element in  $\mathcal{R}(Sp(6, \mathbb{R}))$ , which is  $\theta_3(\pi)$  listed in the following table.*

(2) *If  $\pi = \pi_{-1}(0, 1, \emptyset, 0, 0, (1, 1), (0, 2))$ , then  $\theta_3(\pi) = \pi(0, \emptyset, (1), (1), (1), (2))$ .*

$\pi$	inf. char. of $\theta_3(\pi)$	$\mathcal{A}(\theta_3(\pi))$	$\theta_3(\pi)$	with
$\pi_1((0; 0), -1, \{e_1 \pm f_1\}, 0, 0, 0, 0)$	$(0, 0, 1)$	$\{(1, -1, -1)\}$	$\pi((0), \{-2e_1\}, (1), (1), 0, 0)$	
$\pi_1((0; 0), -1, \{\pm e_1 + f_1\}, 0, 0, 0, 0)$		$\{(1, 1, -1)\}$	$\pi((0), \{2e_1\}, (1), (1), 0, 0)$	
$\pi_1((n; 0), -1, \{e_1 \pm f_1\}, 0, 0, 0, 0)$	$(n, 0, 1)$	$\{(n+1, -1, -1)\}$	$\pi((n, 0, -1), \{e_1 \pm e_2, \pm e_2 - e_3, e_1 \pm e_3, 2e_1, -2e_2, -2e_3\}, 0, 0, 0, 0)$	$1 \leq n \in \mathbb{Z}$
$\pi_1((0; n), -1, \{\pm e_1 + f_1\}, 0, 0, 0, 0)$		$\{(1, 1, -n-1)\}$	$\pi((1, 0, -n), \Psi, \{e_1 \pm e_2, \pm e_2 - e_3, \pm e_1 - e_3, 2e_1, 2e_2, -2e_3\}, 0, 0, 0, 0)$	
$\pi_{-1}(0, 1, \emptyset, 0, 0, (1, -1), (0, \beta))$	$(\beta, 0, 1)$	$\{(1, -1, -1), (1, 1, -1)\}$	$\pi(0, \emptyset, (1), (1), (-1), (\beta))$	$\beta \in \mathbb{C} \setminus \{0\}$
$\pi_{-1}(0, 1, \emptyset, 0, 0, (1, 1), (0, \beta))$		$\{(1, 0, -1)\}$	$\pi(0, \emptyset, (1), (1), (1), (\beta))$	$\beta \in \mathbb{C} \setminus \{\pm 1\}$

*Proof.* (1) can be checked case by case according to Appendix C which lists  $\mathcal{A}(\pi')$  for all  $\pi' \in \mathcal{R}(Sp(6, \mathbb{R}))$  with infinitesimal character  $(\beta, 0, 1)$ .

For (2), now  $\theta_3(\pi)$  has infinitesimal character  $(2, 0, 1)$  and  $\mathcal{A}(\theta_3(\pi)) = \{(1, 0, -1)\}$ . According to Appendix C, there are exactly two elements in  $\mathcal{R}(Sp(6, \mathbb{R}))$  with such infinitesimal character and set of lowest  $K$ -types, which are  $\pi'_1 = \pi(0, \emptyset, (1), (1), (1), (2))$  and  $\pi'_2 = \pi(0, \emptyset, (1), (3), (1), (0))$ . Notice that  $\theta_2(\det_{1,1}) = \pi(0, \emptyset, (1), (1), 0, 0)$  for the determinant character  $\det_{1,1}$  of  $O(1, 1)$  by Proposition 30. So  $\theta_3(\det_{1,1}) = \pi'_1$  by explicit induction principle on  $n$ . Then  $\theta_{2,2}(\pi'_1) \neq 0$  by Kudla's persistence principle.

We claim that  $\theta_2(\theta_{2,2}(\pi'_1)) = 0$ . Otherwise,  $\pi'_1 = \theta_3(\theta_{2,2}(\pi'_1))$  is got from  $\theta_2(\theta_{2,2}(\pi'_1))$  by explicit induction principle on  $n$  (Theorem 19), which must contain entry 1 in  $\kappa$  or 2 in  $\nu$ . But the Langlands parameters of  $\pi'_1$  are not of this form. Therefore  $n(\theta_{2,2}(\pi'_1)) = 3$ . Yet  $\pi'_1$  is not in the list of theta 3-lifts in (1), so  $\theta_{2,2}(\pi'_1) = \pi_{-1}(0, 1, \emptyset, 0, 0, (1, 1), (0, 2))$ .  $\square$

**6.3. For  $O(3, 1)$ .** Let  $\pi = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa) \in \mathcal{R}(O(3, 1))$  with  $n(\pi) = 3$ . By Proposition 34, all these  $\pi$  are listed in the following table.

$\zeta$	$\lambda_d$	$\xi$	$\Psi$	$\mu$	$\nu$	$\varepsilon$	$\kappa$	with
$-1$	$(m; )$	$1$	$\emptyset$	$0$	$0$	$(1)$	$(0)$	$1 \leq m \in \mathbb{Z}$
$1$	$(0; )$	$-1$				$(1)$	$(\beta)$	$\beta \in \mathbb{C} \backslash \{\pm 1\}$
						$(-1)$	$(\beta)$	$\beta \in \mathbb{C}$

*Remark.*  $\det_{3,1} = \pi_1((0; ), -1, \emptyset, 0, 0, (1), (1))$ .

**Proposition 40.** *If  $\pi = \pi_{-1}((m; ), 1, \emptyset, 0, 0, (1), (0))$  with  $1 \leq m \in \mathbb{Z}$ , then*

$$\mathcal{A}(\theta_3(\pi)) = \{(m+1, 2, 0)\}.$$

*Proof.* By Proposition 45 and 46, we get  $\mathcal{A}(\pi) = \{(m; -1) \otimes (; -1)\}$ . So  $\deg(\pi) \leq \deg((m; -1) \otimes (; -1)) = m+2$ . We claim that  $\deg(\pi) = m+2$ . Otherwise,  $\deg(\pi) \leq m$  by Lemma 6 on the parity of degrees. Let  $\delta = (l; \epsilon) \otimes (; \eta) \in \mathcal{D}(\pi)$ . Then  $m \geq \deg(\delta) \geq l$ . But  $||\delta|| = |l+1|^2 > |m+1|^2$ , so  $l > m$ . This makes a contradiction.



Thus  $(m; -1) \otimes (; -1) \in \mathcal{D}(\pi)$ , and hence  $(m+1, 2, 0) \in \mathcal{D}(\theta_3(\pi))$ . Other  $K$ -types for  $O(3, 1)$  of degree  $m+2$  are  $(m+1; +1) \otimes (; -1)$ ,  $(m+1; -1) \otimes (; +1)$ , and  $(m+2; +1) \otimes (; +1)$ . So

$$(m+1, 2, 0) \in \mathcal{D}(\theta_3(\pi)) \subseteq \{(m+1, 2, 0), (m+2, 1, 0), (m+2, 2, 1), (m+3, 1, 1)\}.$$

Since  $\|(m+1, 2, 0)\| < \|(m+2, 1, 0)\| = \|(m+2, 2, 1)\| < \|(m+3, 1, 1)\|$ , by Lemma 17,

$$\mathcal{A}(\theta_3(\pi)) = \{(m+1, 2, 0)\}. \quad \square$$

**Proposition 41.** *Let  $\pi = \pi_1((0), -1, \emptyset, 0, 0, (1), (\beta))$  with  $\beta \in \mathbb{C} \setminus \{\pm 1\}$ . Let  $I$  be the standard module of parabolic induction to get  $\pi$  as in Subsection 2.7.*

(1) *For an  $O(3) \times O(1)$ -type  $\sigma = (l; \epsilon) \otimes (; \eta)$ , the multiplicity of  $\sigma$  in  $I$  is*

$$m(\sigma, I) = \begin{cases} 1, & \text{if } \epsilon = -1 \text{ and } \eta = (-1)^{l+1}, \\ 0, & \text{otherwise.} \end{cases}$$

(2) *The set  $\mathcal{D}(\pi) = \{(1; -1) \otimes (; +1)\}$ , and  $\mathcal{A}(\theta_3(\pi)) = \mathcal{D}(\theta_3(\pi)) = \{(2, 2, 1)\}$ .*

*Proof.* (1) By definition  $I = \text{Ind}_{MAN}^{O(3,1)}(\det \otimes \chi_{1,\beta} \otimes \mathbb{1}_N)$ , where  $MA \cong O(2) \times GL(1, \mathbb{R})$ ,  $\det$  is the determinant character of  $O(2)$ ,  $\chi_{1,\beta}$  is the character  $x \mapsto |x|^\beta$  of  $GL(1, \mathbb{R})$ , and  $\mathbb{1}_N$  is the trivial representation of  $N$ . Notice the  $K$ -intertwining isomorphism  $I|_K \cong \text{Ind}_{K \cap MA}^K(\det \otimes \chi_{1,\beta})$  (cf. [Vog81, Proposition 4.1.12]). By Frobenius reciprocity, the multiplicity of  $\sigma$  in  $I$  is

$$\begin{aligned} m(\sigma, I) &= \dim \text{Hom}_K(\sigma, I|_K) = \dim \text{Hom}_K(\sigma, \text{Ind}_{K \cap MA}^K(\det \otimes \chi_{1,\beta})) \\ &= \dim \text{Hom}_{K \cap MA}(\sigma, \det \otimes \chi_{1,\beta}), \end{aligned}$$

where  $K = O(3) \times O(1)$ , and  $K \cap MA = \{\text{diag}(X, t, t) \mid X \in O(2), t = \pm 1\}$ . Let  $K_0 = K \cap MA$  and  $\lambda = (\det \otimes \chi_{1,\beta})|_{K_0}$ . Then  $\lambda$  is the character of  $K_0$  defined by  $\lambda(\text{diag}(X, t, t)) = \det(X)$ . Write  $V$  for the underlying space of  $\sigma$ . Then

$$m(\sigma, I) = \dim \text{Hom}_{K_0}(V|_{K_0}, \lambda) = \dim V(\lambda),$$

where  $V(\lambda)$  is the  $\lambda$ -isotypic invariant subspace of  $V|_{K_0}$ .

Let us realize  $\sigma$  in a concrete underlying space. Let  $L = \{\text{diag}(Y, 1) \mid Y \in SO(3)\}$ . Then  $L$  is a subgroup of  $O(3) \times O(1)$  isomorphic to  $SO(3)$ , and  $\sigma|_L$  is irreducible and of dimension  $2l+1$ . Recall that there is a unique  $(2l+1)$ -dimensional irreducible representation  $(\sigma_l, V_l)$  of  $SO(3)$ , which can be realized via the double cover

$$\mathbf{P} : SU(2) \twoheadrightarrow SO(3),$$

on the vector space  $V_l$  of polynomials in  $Z_1, Z_2$  that are homogeneous of degree  $2l$ , with actions defined by

$$\left( \sigma_l \left( \mathbf{P} \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \right) f \right) \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = f \left( \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}^{-1} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \right) \quad \text{for } f \in V_l.$$

Here  $f$  is of even degree, so the action is trivial on  $\{\pm Id\} = \ker(\mathbf{P})$ , and  $\sigma_l$  is well defined. Now  $\sigma$  can be realized on  $V_l$ , with actions

$$\begin{aligned} \sigma(\text{diag}(Y, 1)) &= \sigma_l(Y), \\ \sigma(\text{diag}(-1, -1, -1, 1)) &= (-1)^l \epsilon, \\ \sigma(\text{diag}(1, 1, 1, -1)) &= \eta. \end{aligned}$$

The actions of the last two matrices are read off from the signs of  $\sigma$ .

We can find the  $\lambda$ -isotypic invariant subspace  $V(\lambda)$  in  $V = V_l$ . Let

$$r(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \in SO(2).$$

Let  $K_1 = \{\text{diag}(r(\theta), 1, 1)\} \cong SO(2)$ . Since  $K_1$  is a subgroup of  $K_0$  on which  $\lambda$  acts trivially,  $V(\lambda) \subseteq V^{K_1}$  (the subspace of  $K_1$ -fixed vectors in  $V$ ). As

$$\mathbf{P} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = r(\theta),$$

it is easy to see that  $V^{K_1} = \mathbb{C}Z_1^l Z_2^l$ , which is of dimension 1. Notice that  $K_0$  is generated by  $K_1$ ,  $\text{diag}(1, -1, 1, 1)$ , and  $\text{diag}(-1, -1, -1, -1)$ . So  $V(\lambda) = \mathbb{C}Z_1^l Z_2^l$  or 0, depending on whether or not the following two equations hold:

$$\begin{aligned} \sigma|_{\mathbb{C}Z_1^l Z_2^l}(\text{diag}(1, -1, 1, 1)) &= \lambda(\text{diag}(1, -1, 1, 1)) = -1, \\ \sigma|_{\mathbb{C}Z_1^l Z_2^l}(\text{diag}(-1, -1, -1, -1)) &= \lambda(\text{diag}(-1, -1, -1, -1)) = 1. \end{aligned}$$

Notice that  $\text{diag}(1, -1, 1, 1) = \text{diag}(-1, -1, -1, 1) \cdot \text{diag}(-1, 1, -1, 1)$ . Since

$$\mathbf{P} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \text{diag}(-1, 1, -1) \in SO(3),$$

$\sigma|_{\mathbb{C}Z_1^l Z_2^l}(\text{diag}(-1, 1, -1, 1)) = (-1)^l$ , and thus

$$\sigma|_{\mathbb{C}Z_1^l Z_2^l}(\text{diag}(1, -1, 1, 1)) = (-1)^l \epsilon (-1)^l = \epsilon.$$

Moreover,

$$\sigma(\text{diag}(-1, -1, -1, -1)) = \sigma(\text{diag}(-1, -1, -1, 1)) \cdot \sigma(\text{diag}(1, 1, 1, -1)) = (-1)^l \epsilon \eta.$$

So  $V(\lambda) = \begin{cases} \mathbb{C}Z_1^l Z_2^l, & \text{if } \epsilon = -1 \text{ and } (-1)^l \epsilon \eta = 1, \\ 0, & \text{otherwise.} \end{cases}$

(2) By Proposition 45 and 46, we get  $\mathcal{A}(\pi) = \{\Lambda\}$  with  $\Lambda = (0; -1) \otimes (; -1)$ . So  $\deg(\pi) \leq \deg(\Lambda) = 4$ . Since  $\phi_3(\Lambda) = 0$ , we have  $\Lambda \notin \mathcal{D}(\pi)$ . By Lemma 6 on the parity of degrees,  $\deg(\pi) = 2$  or 0. The only  $K$ -type for  $O(3, 1)$  of degree 0 is  $(0; +1) \otimes (; +1)$ , which cannot occur in  $\pi$  since it has the same norm as that of the lowest  $K$ -type of  $\pi$ . So  $\deg(\pi) = 2$ .

All  $K$ -types for  $O(3, 1)$  of degree 2 are  $(1; +1) \otimes (; -1)$ ,  $(1; -1) \otimes (; +1)$ , and  $(2; +1) \otimes (; +1)$ . By (1), we see that  $(1; +1) \otimes (; -1)$  and  $(2; +1) \otimes (; +1)$  do not occur in  $I$ , and thus not in  $\pi$ . So  $\mathcal{D}(\pi) = \{(1; -1) \otimes (; +1)\}$ . So  $\mathcal{A}(\theta_3(\pi)) = \mathcal{D}(\theta_3(\pi)) = \{(2, 2, 1)\}$  by Lemma 17.  $\square$

**Proposition 42.** *Let  $\pi = \pi_1((0), -1, \emptyset, 0, 0, (-1), (\beta))$  with  $\beta \in \mathbb{C}$ . Let  $I$  be the standard module of parabolic induction to get  $\pi$  as in Subsection 2.7.*

(1) *For an  $O(3) \times O(1)$ -type  $\sigma = (l; \epsilon) \otimes (; \eta)$ , the multiplicity of  $\sigma$  in  $I$  is*

$$m(\sigma, I) = \begin{cases} 1, & \text{if } \epsilon = -1 \text{ and } \eta = (-1)^l, \\ 0, & \text{otherwise.} \end{cases}$$

(2)  $\mathcal{A}(\theta_3(\pi)) = \{(2, 2, 2)\}$ .

*Proof.* (1) Now  $I = \text{Ind}_{MAN}^{O(3,1)}(\det \otimes \chi_{-1,\beta} \otimes \mathbb{1}_N)$ , where  $MA \cong O(2) \times GL(1, \mathbb{R})$ ,  $\det$  is the determinant character of  $O(2)$ ,  $\chi_{-1,\beta}$  is the character  $x \mapsto \text{sgn}(x)|x|^\beta$  of  $GL(1, \mathbb{R})$ , and  $\mathbb{1}_N$  is the

trivial representation of  $N$ . Notice the  $K$ -intertwining isomorphism  $I|_K \cong \text{Ind}_{K \cap MA}^K(\det \otimes \chi_{-1, \beta})$  (cf. [Vog81, Proposition 4.1.12]). By Frobenius reciprocity, the multiplicity of  $\sigma$  in  $I$  is

$$\begin{aligned} m(\sigma, I) &= \dim \text{Hom}_K(\sigma, I|_K) = \dim \text{Hom}_K(\sigma, \text{Ind}_{K \cap MA}^K(\det \otimes \chi_{-1, \beta})) \\ &= \dim \text{Hom}_{K \cap MA}(\sigma, \det \otimes \chi_{-1, \beta}), \end{aligned}$$

where  $K = O(3) \times O(1)$ , and  $K \cap MA = \{\text{diag}(X, t, t) \mid X \in O(2), t = \pm 1\}$ . Let  $K_0 = K \cap MA$ ,  $\lambda' = (\det \otimes \chi_{-1, \beta})|_{K_0}$ . Then  $\lambda'$  is the character of  $K_0$  defined by  $\lambda'(\text{diag}(X, t, t)) = \det(X)t$ . Write  $V$  for the underlying space of  $\sigma$ . Then

$$m(\sigma, I) = \dim_{K_0} \text{Hom}(V|_{K_0}, \lambda') = \dim V(\lambda'),$$

where  $V(\lambda')$  is the  $\lambda'$ -isotypic invariant subspace of  $V|_{K_0}$ . By a similar argument as that in the proof of last proposition, replacing  $\lambda$  by  $\lambda'$ , we get

$$V(\lambda) = \begin{cases} \mathbb{C}Z_1^l Z_2^l, & \text{if } \epsilon = -1 \text{ and } (-1)^{l+1} \epsilon \eta = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(2) By Proposition 45 and 46,  $\mathcal{A}(\pi) = \{\Lambda\}$ , where  $\Lambda = (0; -1) \otimes (; +1)$ . So  $\deg(\pi) \leq \deg(\Lambda) = 3$ . By Lemma 6 on the parity of degrees,  $\deg(\pi) = 1$  or 3. List all  $O(3) \times O(1)$ -types of degree 1 or 3:  $(0; +1) \otimes (; -1)$ ,  $(1; +1) \otimes (; +1)$ ,  $(2; +1) \otimes (; -1)$ ,  $(3; +1) \otimes (; +1)$ ,  $(0; -1) \otimes (; +1)$ ,  $(1; -1) \otimes (; -1)$ ,  $(2; -1) \otimes (; +1)$ . In this list, the first two are of degree 1, and the left are of degree 3.

By (1), the first four  $O(3) \times O(1)$ -types in this list do not occur in  $\pi$ . So  $\deg(\pi) = 3$  and

$$\begin{aligned} (0; -1) \otimes (; +1) &\in \mathcal{D}(\pi) \subseteq \{(0; -1) \otimes (; +1), (1; -1) \otimes (; -1), (2; -1) \otimes (; +1)\}, \\ (2, 2, 2) &\in \mathcal{D}(\theta_3(\pi)) \subseteq \{(2, 2, 2), (2, 2, 0), (3, 2, 1)\}, \end{aligned}$$

with norms  $\|(2, 2, 2)\| < \|(2, 2, 0)\| < \|(3, 2, 1)\|$ . By Lemma 17, the set  $\mathcal{A}(\theta_3(\pi))$  consists of  $U(3)$ -types with minimal norm in  $\mathcal{D}(\theta_3(\pi))$ , so  $\mathcal{A}(\theta_3(\pi)) = \{(2, 2, 2)\}$ .  $\square$

**Theorem 43.** *Let  $\pi \in \mathcal{R}(O(3, 1))$  with  $n(\pi) = 3$ . Then the infinitesimal character of  $\theta_3(\pi)$  and the set  $\mathcal{A}(\theta_3(\pi))$  determine a unique element in  $\mathcal{R}(Sp(6, \mathbb{R}))$ , which is  $\theta_3(\pi)$  listed in the following table.*

$\pi$	inf.char. of $\theta_3(\pi)$	$\mathcal{A}(\theta_3(\pi))$	$\theta_3(\pi)$	with
$\pi_{-1}((m; ), 1, \emptyset, 0, 0, (1), (0))$	$(m, 0, 1)$	$\{(m+1, 2, 0)\}$	$\pi((m), \{2e_1\}, (1), (1), 0, 0)$	$1 \leq m \in \mathbb{Z}$
$\pi_1((0; ), -1, \emptyset, 0, 0, (1), (0))$	$(0, 0, 1)$	$\{(2, 2, 1)\}$	$\pi((1, 0, 0), \{e_1 \pm e_2, e_2 \pm e_3, e_1 \pm e_3, 2e_1, 2e_2, -2e_3\}, 0, 0, 0, 0)$	
$\pi_1((0; ), -1, \emptyset, 0, 0, (1), (\beta))$	$(\beta, 0, 1)$		$\pi((1, 0), \{2e_1, 2e_2, e_1 \pm e_2\}, 0, 0, (-1), (\beta))$	$\beta \in \mathbb{C} \setminus \{0, \pm 1\}$
$\pi_1((0), -1, \emptyset, 0, 0, (-1), (\beta))$		$\{(2, 2, 2)\}$	$\pi((1, 0), \{2e_1, 2e_2, e_1 \pm e_2\}, 0, 0, (1), (\beta))$	$\beta \in \mathbb{C}$

*Proof.* This can be checked case by case according to Appendix C which lists  $\mathcal{A}(\pi')$  for all  $\pi' \in \mathcal{R}(Sp(6, \mathbb{R}))$  with infinitesimal character  $(\beta, 0, 1)$ .  $\square$

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## APPENDIX A. ALGORITHMS TO COMPUTE LOWEST $K$ -TYPES

In this appendix, we quote the algorithms from [Pau05] for  $\mathcal{R}(Sp(2n, \mathbb{R}))$  and  $\mathcal{R}(O(p, q))$  (with  $p + q$  even) to compute the set of lowest  $K$ -types from the Langlands parameters.

### A.1. For $Sp(2n, \mathbb{R})$ .

**Proposition 44** ([Pau05, Proposition 6]). *Let  $\pi = \pi(\lambda_d, \Psi, \mu, \nu, \varepsilon, \kappa) \in \mathcal{R}(Sp(2n, \mathbb{R}))$ ,  $\lambda_d =$*

$$\underbrace{(a_1, \dots, a_1)}_{k_1}, \underbrace{(a_2, \dots, a_2)}_{k_2}, \dots, \underbrace{(a_b, \dots, a_b)}_{k_b}, \underbrace{(0, \dots, 0)}_z, \underbrace{(-a_b, \dots, -a_b)}_{l_b}, \dots, \underbrace{(-a_2, \dots, -a_2)}_{l_2}, \underbrace{(-a_1, \dots, -a_1)}_{l_1}$$

*with integers  $a_1 > a_2 > \dots > a_b > 0$ , and  $|k_i - l_i| \leq 1$  for all  $i$ . Let  $k = \sum_{i=1}^b k_i$ ,  $l = \sum_{i=1}^b l_i$ ,  $\tilde{k}_j = \sum_{i=1}^j k_i$ , and  $\tilde{l}_j = \sum_{i=1}^j l_i$ . Notice that  $k + z + l = v$ .*

*Let  $\lambda_a$  be obtained from  $(\lambda_d \mid \frac{\mu_1}{2}, \dots, \frac{\mu_s}{2}, \underbrace{0, \dots, 0}_t, -\frac{\mu_s}{2}, \dots, -\frac{\mu_1}{2})$  by reordering of the coordinates so the resulting entries are nonincreasing. Write  $\lambda_a =$*

$$\underbrace{(\alpha_1, \dots, \alpha_1)}_{u_1}, \underbrace{(\alpha_2, \dots, \alpha_2)}_{u_2}, \dots, \underbrace{(\alpha_m, \dots, \alpha_m)}_{u_m}, \underbrace{(0, \dots, 0)}_w, \underbrace{(-\alpha_m, \dots, -\alpha_m)}_{r_m}, \dots, \underbrace{(-\alpha_2, \dots, -\alpha_2)}_{r_2}, \underbrace{(-\alpha_1, \dots, -\alpha_1)}_{r_1}$$

*with  $\alpha_1 > \alpha_2 > \dots > \alpha_m > 0$ . Then we have for all  $i$  that  $|u_i - r_i| \leq 1$ ,  $\alpha_i \in \frac{1}{2}\mathbb{Z}$ . If  $u_i \neq r_i$ , then  $\alpha_i \in \mathbb{Z}$ . Let  $u = \sum_{i=1}^m u_i$  and  $r = \sum_{i=1}^m r_i$ . Notice that  $u - r = k - l$ .*

*For  $Sp(2n, \mathbb{R})$ , the root system is*

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm 2e_i \mid 1 \leq i \leq n\}.$$

*We fix the standard set of positive compact roots*

$$\Delta_c^+ = \{e_i - e_j \mid 1 \leq i < j \leq n\}.$$

*Let  $\rho(\mathfrak{u} \cap \mathfrak{p})$  and  $\rho(\mathfrak{u} \cap \mathfrak{k})$  be one-half sums of the noncompact and compact roots with respect to which  $\lambda_a$  is strictly dominant, respectively. Then  $\lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k})$*

$$\begin{aligned} &= (\underbrace{\beta_1, \dots, \beta_1}_{u_1}, \dots, \underbrace{\beta_m, \dots, \beta_m}_{u_m}, \underbrace{u - r, \dots, u - r}_w, \underbrace{\gamma_m, \dots, \gamma_m}_{r_m}, \dots, \underbrace{\gamma_1, \dots, \gamma_1}_{r_1}), \\ &\beta_i = \alpha_i + \frac{1}{2} + \frac{u_i - r_i}{2} + \sum_{1 \leq j < i} (u_j - r_j), \\ &\gamma_i = -\alpha_i - \frac{1}{2} + \frac{u_i - r_i}{2} + \sum_{1 \leq j < i} (u_j - r_j). \end{aligned}$$

*Then the lowest  $K$ -types of  $\pi$  are precisely those of form*

$$\begin{aligned} \Lambda &= \lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) + \delta_L, \\ \text{with } \delta_L &= (\underbrace{\delta_1, \dots, \delta_1}_{u_1}, \dots, \underbrace{\delta_m, \dots, \delta_m}_{u_m}, \eta_1, \dots, \eta_w, \underbrace{\delta_m, \dots, \delta_m}_{r_m}, \dots, \underbrace{\delta_1, \dots, \delta_1}_{r_1}), \end{aligned}$$

*satisfying the following conditions:*

- (1) *If  $\beta_i \in \mathbb{Z}$ , then  $\delta_i = 0$ .*

(2) Suppose  $\beta_i \in \mathbb{Z} + \frac{1}{2}$ . Then  $\delta_i \in \{\pm \frac{1}{2}\}$ . If  $\alpha_i$  does not occur as an entry in  $\lambda_d$  then both choices occur. If  $\alpha_i = a_j$ , then  $\delta_i = \frac{1}{2}$  if  $e_{\tilde{k}_{j-1}+1} + e_{v-\tilde{l}_j+1} \in \Psi$ , and  $\delta_i = -\frac{1}{2}$  otherwise.

(3) Let  $h = \#\{j \mid \varepsilon_j = (-1)^{u-r+1}\} + \#\{j \mid \mu_j = 0\} + \lceil \frac{z+1}{2} \rceil$ . Then  $(\eta_1, \dots, \eta_w) = (\underbrace{1, \dots, 1}_h, 0, \dots, 0)$  or  $(0, \dots, 0, \underbrace{-1, \dots, -1}_h)$ . If  $z = 0$  then both choices occur. If  $z > 0$ , then  $(\eta_1, \dots, \eta_w)$  is of the first form whenever  $e_{k+1} + e_{k+z} \in \Psi$  (this includes the case  $z = 1$  where the condition becomes  $2e_{k+1} \in \Psi$ ), and of the second form otherwise.

## A.2. For $O(p, q)$ with $p + q$ even.

**Proposition 45** ([Pau05, Proposition 10]). Let  $\pi = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa) \in \mathcal{R}(O(p, q))$  with  $p + q$  even. Let  $\lambda_d =$

$$(\underbrace{a_1, \dots, a_1}_{k_1}, \underbrace{a_2, \dots, a_2}_{k_2}, \dots, \underbrace{a_b, \dots, a_b}_{k_b}, \underbrace{0, \dots, 0}_z; \underbrace{a_1, \dots, a_1}_{l_1}, \underbrace{a_2, \dots, a_2}_{l_2}, \dots, \underbrace{a_b, \dots, a_b}_{l_b}, \underbrace{0, \dots, 0}_{z'})$$

with integers  $a_1 > a_2 > \dots > a_b > 0$ ,  $|k_i - l_i| \leq 1$  for all  $i$ , and  $|z - z'| \leq 1$ . Let  $k = \sum_{i=1}^b k_i$ ,  $l = \sum_{i=1}^b l_i$ ,  $\tilde{k}_j = \sum_{i=1}^j k_i$ , and  $\tilde{l}_j = \sum_{i=1}^j l_i$ .

Let  $\lambda_a$  be obtained from  $(\lambda_d \mid (\underbrace{\frac{\mu_1}{2}, \dots, \frac{\mu_z}{2}}_{[\frac{z}{2}]}, \underbrace{0, \dots, 0}_{[\frac{z}{2}]}; \underbrace{\frac{\mu_1}{2}, \dots, \frac{\mu_z}{2}}_{[\frac{z'}{2}]}, \underbrace{0, \dots, 0}_{[\frac{z'}{2}]})$ , by reordering of the coordinates so the resulting entries in both parts are nonincreasing. Write  $\lambda_a =$

$$(\underbrace{\alpha_1, \dots, \alpha_1}_{u_1}, \underbrace{\alpha_2, \dots, \alpha_2}_{u_2}, \dots, \underbrace{\alpha_m, \dots, \alpha_m}_{u_m}, \underbrace{0, \dots, 0}_x; \underbrace{\alpha_1, \dots, \alpha_1}_{r_1}, \underbrace{\alpha_2, \dots, \alpha_2}_{r_2}, \dots, \underbrace{\alpha_m, \dots, \alpha_m}_{r_m}, \underbrace{0, \dots, 0}_y)$$

with  $\alpha_1 > \alpha_2 > \dots > \alpha_m > 0$ . Then we have for all  $i$  that  $|u_i - r_i| \leq 1$ ,  $|x - y| \leq 1$ ,  $\alpha_i \in \frac{1}{2}\mathbb{Z}$ . If  $u_i \neq r_i$ , then  $\alpha_i \in \mathbb{Z}$ . Let  $u = \sum_{i=1}^m u_i$  and  $r = \sum_{i=1}^m r_i$ . Notice that  $u - r = k - l$  and  $x - y = z - z'$ .

We take the set of roots  $\Delta$  for  $O(p, q)$ , and fix the standard set of positive compact roots  $\Delta_c^+$ , which are as in the following table (with  $p_0 = \lceil \frac{p}{2} \rceil$  and  $q_0 = \lceil \frac{q}{2} \rceil$ ):

	$p, q$ are both even	$p, q$ are both odd
$\Delta$	$\{\pm e_i \pm e_j \mid 1 \leq i < j \leq p_0\}$ $\cup \{\pm f_i \pm f_j \mid 1 \leq i < j \leq q_0\}$ $\cup \{\pm e_i \pm f_j \mid 1 \leq i \leq p_0, 1 \leq j \leq q_0\}$	$\{\pm e_i \pm e_j \mid 1 \leq i < j \leq p_0\}$ $\cup \{\pm f_i \pm f_j \mid 1 \leq i < j \leq q_0\}$ $\cup \{\pm e_i \pm f_j, \pm e_i, \pm f_j \mid 1 \leq i \leq p_0, 1 \leq j \leq q_0\}$
$\Delta_c^+$	$\{e_i \pm e_j \mid 1 \leq i < j \leq p_0\}$ $\cup \{\pm f_i \pm f_j \mid 1 \leq i < j \leq q_0\}$	$\{e_i \pm e_j \mid 1 \leq i < j \leq p_0\}$ $\cup \{f_i \pm f_j \mid 1 \leq i < j \leq q_0\}$ $\cup \{e_i, f_j \mid 1 \leq i \leq p_0, 1 \leq j \leq q_0\}$

Let  $\rho(\mathfrak{u} \cap \mathfrak{p})$  and  $\rho(\mathfrak{u} \cap \mathfrak{k})$  be one-half sums of the noncompact and compact roots with respect to which  $\lambda_a$  is strictly dominant, respectively. Then  $\lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k})$

$$= (\underbrace{\beta_1, \dots, \beta_1}_{u_1}, \dots, \underbrace{\beta_m, \dots, \beta_m}_{u_m}, \underbrace{0, \dots, 0}_{x'}; \underbrace{\gamma_1, \dots, \gamma_1}_{r_1}, \dots, \underbrace{\gamma_m, \dots, \gamma_m}_{r_m}, \underbrace{0, \dots, 0}_y),$$

$$\beta_i = \alpha_i + \frac{1}{2} - u + r + \frac{u_i - r_i}{2} + \sum_{1 \leq j < i} (u_j - r_j),$$

$$\gamma_i = -\alpha_i - \frac{1}{2} - u + r + \frac{u_i - r_i}{2} + \sum_{1 \leq j < i} (u_j - r_j).$$

Then the highest weights of the lowest  $K$ -types of  $\pi$  are precisely those of form

$$\Lambda_0 = \lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) + \delta_L,$$

with  $\delta_L = (\underbrace{\delta_1, \dots, \delta_1}_{u_1}, \dots, \underbrace{\delta_m, \dots, \delta_m}_{u_m}, \eta_1, \dots, \eta_x; \underbrace{-\delta_1, \dots, -\delta_1}_{r_1}, \dots, \underbrace{-\delta_m, \dots, -\delta_m}_{r_m}, \xi_1, \dots, \xi_y),$

satisfying the following conditions:

- (1) If  $\beta_i \in \mathbb{Z}$ , then  $\delta_i = 0$ .
- (2) Suppose  $\beta_i \in \mathbb{Z} + \frac{1}{2}$ . Then  $\delta_i \in \{\pm \frac{1}{2}\}$ . If  $\alpha_i$  does not occur as an entry in  $\lambda_d$  then both choices occur. If  $\alpha_i = a_j$ , then  $\delta_i = \frac{1}{2}$  if  $e_{\tilde{k}_j} - f_{\tilde{l}_j} \in \Psi$ , and  $\delta_i = -\frac{1}{2}$  otherwise.
- (3) Let  $h = \min\{z, z'\} + \#\{j \mid \mu_j = 0\} + \min\{\beta, \gamma\}$ , where  $\beta = \#\{j \mid \varepsilon_j = 1\}$  and  $\gamma = \#\{j \mid \varepsilon_j = -1\}$ . Then

$$(\eta_1, \dots, \eta_x; \xi_1, \dots, \xi_y) = (\underbrace{1, \dots, 1}_h, 0, \dots, 0; 0, \dots, 0) \text{ or } (0, \dots, 0; \underbrace{1, \dots, 1}_h, 0, \dots, 0).$$

If  $z + z' = 0$  then both choices occur. If  $z + z' > 0$ , then  $(\eta_1, \dots, \eta_w)$  is of the first form whenever  $e_{k+z} - f_{l+z'} \in \Psi$ , and of the second form otherwise.

**Proposition 46** ([Pau05, Proposition 13]). Let  $\pi = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa) \in \mathcal{R}(O(p, q))$  with  $p+q$  even. Let  $\Lambda_0 = (\Lambda_1, \Lambda_2)$  be the highest weight of a lowest  $K$ -type of  $\pi$  as in Proposition 45. Let  $z, z', \beta$  and  $\alpha$  as in Proposition 45, and  $\alpha = \#\{j \mid \mu_j = 0\}$ . Then  $\pi$  has a lowest  $K$ -type  $\Lambda$  with highest  $\Lambda_0$  and signs given as follows:

- (1) Suppose  $z + z' = 0$  and  $\kappa_i \neq 0$  for all  $i$ . If  $\beta \geq \gamma$  then both  $(1; 1)$  and  $(-1; -1)$  occur as signs. (The resulting two  $K$ -types may coincide.) If  $\beta < \gamma$  then both  $(1; -1)$  and  $(-1; 1)$  occur as signs.
- (2) Suppose  $z + z' = 0$  and  $(\varepsilon_i, \kappa_i) = (1, 0)$  for some  $i$ . If  $\beta \geq \gamma$  then the signs are  $(\zeta; \zeta)$ . If  $\beta < \gamma$  then the signs are  $(\zeta; -\zeta)$  if  $\Lambda_1$  has more zeros than  $\Lambda_2$ , and  $(-\zeta; \zeta)$  otherwise.
- (3) Suppose  $z + z' = 0$  and  $(\varepsilon_i, \kappa_i) = (-1, 0)$  for some  $i$ . If  $\beta \geq \gamma$  then the signs are  $(\zeta; \zeta)$  if  $\Lambda_1$  has more zeros than  $\Lambda_2$ , and  $(-\zeta, -\zeta)$  otherwise. If  $\beta < \gamma$ , then the signs are  $(\zeta; -\zeta)$ .
- (4) Suppose  $z + z' > 0$  and  $\beta \geq \gamma$ . Then the signs are  $(\xi; \xi)$ .
- (5) Suppose  $z + z' > 0$  and  $\beta < \gamma$ . Then the signs are  $(\xi; -\xi)$  if  $\Lambda_1$  has more zeros than  $\Lambda_2$ , and  $(-\xi; \xi)$  otherwise.

## APPENDIX B. THETA 1-LIFTS AND 2-LIFTS WHEN $p + q = 4$

For the sake of completeness, in this appendix we list all nonzero theta 1-lifts and 2-lifts for  $O(p, q)$  when  $(p, q) = (4, 0), (3, 1), (2, 2)$ , which is read off from [Pau05, Theorem 15, 18].

**B.1. For  $O(4, 0)$ .** Let  $\pi = \pi_1((m, l; ), \xi, \{e_1 \pm e_2\}, 0, 0, 0, 0) \in \mathcal{R}(O(4, 0))$ , with integers  $m > l \geq 0$ . All nonzero theta 1-lifts and 2-lifts of  $\pi$  are in the following table:

	$m$	$l$	$\xi$	
$\theta_1(\pi) \neq 0$	$\geq 1$	0	1	$\theta_1(\pi) = \pi((m), \{2e_1\}, 0, 0, 0, 0)$
$\theta_2(\pi) \neq 0$	$> l$	$\geq 0$		$\theta_2(\pi) = \pi((m, l), \{e_1 \pm e_2, 2e_1, 2e_2\}, 0, 0, 0, 0)$

**B.2. For  $O(3, 1)$ .** Let  $\pi = \pi_\zeta((m;), \xi, \emptyset, 0, 0, \varepsilon, \kappa) \in \mathcal{R}(O(3, 1))$  with  $0 \leq m \in \mathbb{Z}$ ,  $\varepsilon = (\varepsilon_1)$  and  $\kappa = (\kappa_1)$ . All  $\pi$  with  $\theta_1(\pi) \neq 0$  are in the following table:

$\zeta$	$m$	$\xi$	$(\varepsilon_1, \kappa_1)$	$\theta_1(\pi)$
1	$\geq 0$	1	$(1, 0)$	$\pi((m), \{2e_1\}, 0, 0, 0, 0)$
	0		$\neq (1, 0)$	$\pi(0, \emptyset, 0, 0, -\varepsilon, \kappa)$

All  $\pi$  with  $\theta_2(\pi) \neq 0$  are in the following table:

$\zeta$	$m$	$\xi$	$(\varepsilon_1, \kappa_1)$	$\theta_2(\pi)$
1	$\geq 0$	1	$\neq (-1, 0)$	$\pi((m), \{2e_1\}, 0, 0, -\varepsilon, \kappa)$
			$(-1, 0)$	$\pi((m, 0), \{e_1 \pm e_2, 2e_1, -2e_2\}, 0, 0, 0, 0)$
-1	$\geq 1$		$(-1, 0)$	$\pi((m, 0), \{e_1 \pm e_2, 2e_1, 2e_2\}, 0, 0, 0, 0)$

**B.3. For  $O(2, 2)$ .** Let  $\pi = \pi_\zeta(\lambda_d, \xi, \Psi, \mu, \nu, \varepsilon, \kappa) \in \mathcal{R}(O(2, 2))$ . All  $\pi$  with  $\theta_1(\pi) \neq 0$  are in the following table:

$\zeta$	$\lambda_d$	$\xi$	$\Psi$	$\mu$	$\nu$	$\varepsilon$	$\kappa$	$\theta_1(\pi)$	with
1	$(m; 0)$	1	$\{e_1 \pm f_1\}$	0	0	0	0	$\pi((m), \{2e_1\}, 0, 0, 0, 0)$	$0 \leq m \in \mathbb{Z}$
	$(0; m)$		$\{\pm e_1 + f_1\}$					$\pi((-m), \{-2e_1\}, 0, 0, 0, 0)$	
	0		$\emptyset$			$(1, \eta)$	$(0, \beta)$	$\pi(0, \emptyset, 0, 0, (\eta), (\beta))$	$(\eta, \beta) \neq (-1, 0)$

All  $\pi$  with  $\theta_2(\pi) \neq 0$  are in the following table:

$\zeta$	$\lambda_d$	$\xi$	$\Psi$	$\mu$	$\nu$	$\varepsilon$	$\kappa$	$\theta_2(\pi)$	with
1	$(m; l)$	1	$\{e_1 \pm f_1\}$	0	0	0	0	$\pi((m, -l), \{e_1 \pm e_2, 2e_1, -2e_2\}, 0, 0, 0, 0)$	integers $m \geq l \geq 0$
			$\{\pm e_1 + f_1\}$					$\pi((m, -l), \{\pm e_1 - e_2, 2e_1, -2e_2\}, 0, 0, 0, 0)$	integers $l \geq m \geq 0$
	$\emptyset$		$(\mu_1)$	$(\nu_1)$	$\pi(0, \emptyset, (\mu_1), (\nu_1), 0, 0)$			$0 \leq \mu_1 \in \mathbb{Z}, \nu_1 \in \mathbb{C}$ $\nu_1 = 0 \Rightarrow \mu_1$ is odd	
			0	0	$(\varepsilon_1, \varepsilon_2)$	$(\kappa_1, \kappa_2)$	$\pi(0, \emptyset, 0, 0, (\varepsilon_1, \varepsilon_2), (\kappa_1, \kappa_2))$	$(\varepsilon_i, \kappa_i) \neq (-1, 0)$ for $i = 1, 2$	
$\pi((0), \{-2e_1\}, 0, 0, (\varepsilon_2), (\kappa_2))$	$(\varepsilon_1, \kappa_1) = (-1, 0)$ $(\varepsilon_2, \kappa_2) \neq (1, 0)$								
$\pi((0), \{2e_1\}, 0, 0, (\varepsilon_2), (\kappa_2))$									
-1									

## APPENDIX C. REPRESENTATIONS OF $Sp(6, \mathbb{R})$ WITH INFINITESIMAL CHARACTER $(\beta, 0, 1)$

In this appendix, we list all  $\pi \in \mathcal{R}(Sp(6, \mathbb{R}))$  with infinitesimal character  $(\beta, 0, 1)$  (for any  $\beta \in \mathbb{C}$ ) and calculate  $\mathcal{A}(\pi)$  case by case. Let  $\pi = \pi(\lambda_d, \Psi, \mu, \nu, \varepsilon, \kappa)$ , with  $MA \cong Sp(2v, \mathbb{R}) \times GL(2, \mathbb{R})^s \times GL(1, \mathbb{R})^t$ . As  $v + 2s + t = 3$ ,  $(v, s, t) = (3, 0, 0), (2, 0, 1), (1, 0, 2), (0, 0, 3), (0, 1, 1)$ , or  $(1, 1, 0)$ . Let  $\sim$  denote the equivalence up to permutations and sign-changes of coordinates. Without loss of generality, we assume that  $\beta, \nu_i$  and  $\kappa_j$  all lie in  $\{z \in \mathbb{C} \mid \Re(z) \geq 0\}$ .



C.1.  $(\mathbf{v}, \mathbf{s}, \mathbf{t}) = (\mathbf{3}, \mathbf{0}, \mathbf{0})$ .  $\lambda_d \sim (0, 0, 1)$ ,  $(1, 0, 1)$  or  $(\beta, 0, 1)$  with  $2 \leq \beta \in \mathbb{Z}$ .

$\lambda_d = \lambda_a$	$\frac{\rho(\mathbf{u} \cap \mathfrak{p})}{-\rho(\mathbf{u} \cap \mathfrak{k})}$	$\Psi$	$\mathcal{A}(\pi)$
$(1, 0, 0)$	$(1, 1, 1)$	$\{e_1 \pm e_2, e_2 \pm e_3, e_1 \pm e_3, 2e_1, 2e_2, -2e_3\}$	$\{(2, 2, 1)\}$
		$\{e_1 \pm e_2, \pm e_2 - e_3, e_1 \pm e_3, 2e_1, 2e_2, -2e_3\}$	$\{(2, 1, 0)\}$
$(0, 0, -1)$	$(-1, -1, -1)$	$\{e_1 \pm e_2, \pm e_2 - e_3, \pm e_1 - e_3, 2e_1, -2e_2, -2e_3\}$	$\{(0, -1, -2)\}$
		$\{\pm e_1 - e_2, \pm e_2 - e_3, \pm e_1 - e_3, 2e_1, -2e_2, 2e_3\}$	$\{(-1, -2, -2)\}$
$(1, 0, -1)$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$\{e_1 \pm e_2, \pm e_2 - e_3, e_1 \pm e_3, 2e_1, 2e_2, -2e_3\}$	$\{(2, 1, -1)\}$
		$\{e_1 \pm e_2, \pm e_2 - e_3, e_1 \pm e_3, 2e_1, -2e_2, -2e_3\}$	$\{(2, -1, -1)\}$
		$\{e_1 \pm e_2, \pm e_2 - e_3, \pm e_1 - e_3, 2e_1, 2e_2, -2e_3\}$	$\{(1, 1, -2)\}$
		$\{e_1 \pm e_2, \pm e_2 - e_3, \pm e_1 - e_3, 2e_1, -2e_2, -2e_3\}$	$\{(1, -1, -2)\}$
$(\beta, 1, 0)$	$(1, 2, 2)$	$\{e_1 \pm e_2, e_2 \pm e_3, e_1 \pm e_3, 2e_1, 2e_2, 2e_3\}$	$\{(\beta + 1, 3, 3)\}$
		$\{e_1 \pm e_2, e_2 \pm e_3, e_1 \pm e_3, 2e_1, 2e_2, -2e_3\}$	$\{(\beta + 1, 3, 1)\}$
$(0, -1, -\beta)$	$(-2, -2, -1)$	$\{\pm e_1 - e_2, \pm e_2 - e_3, \pm e_1 - e_3, 2e_1, -2e_2, -2e_3\}$	$\{(-1, -3, -\beta - 1)\}$
		$\{\pm e_1 - e_2, \pm e_2 - e_3, \pm e_1 - e_3, -2e_1, -2e_2, -2e_3\}$	$\{(-3, -3, -\beta - 1)\}$
$(\beta, 0, -1)$	$(1, 0, 0)$	$\{e_1 \pm e_2, \pm e_2 - e_3, e_1 \pm e_3, 2e_1, 2e_2, -2e_3\}$	$\{(\beta + 1, 1, -1)\}$
		$\{e_1 \pm e_2, \pm e_2 - e_3, e_1 \pm e_3, 2e_1, -2e_2, -2e_3\}$	$\{(\beta + 1, -1, -1)\}$
$(1, 0, -\beta)$	$(0, 0, -1)$	$\{e_1 \pm e_2, \pm e_2 - e_3, \pm e_1 - e_3, 2e_1, 2e_2, -2e_3\}$	$\{(1, 1, -\beta - 1)\}$
		$\{e_1 \pm e_2, \pm e_2 - e_3, \pm e_1 - e_3, 2e_1, -2e_2, -2e_3\}$	$\{(1, -1, -\beta - 1)\}$

C.2.  $(\mathbf{v}, \mathbf{s}, \mathbf{t}) = (\mathbf{2}, \mathbf{0}, \mathbf{1})$ .  $\varepsilon = (\varepsilon_1)$ ,  $\kappa = (\kappa_1)$ , and  $(\lambda_d \mid \kappa) \sim (\beta, 0, 1)$ .

(F-2):  $(\varepsilon_1, \kappa_1) \neq (-1, 0)$ .

(1)  $\kappa \sim (\beta)$ ,  $\lambda_d \sim (1, 0)$ , and  $(\varepsilon_1, \beta) \neq (-1, 0)$ .

$\kappa_1$	$\lambda_d$	$\lambda_a$	$\frac{\rho(\mathbf{u} \cap \mathfrak{p})}{-\rho(\mathbf{u} \cap \mathfrak{k})}$	$\Psi$	$\varepsilon_1$	$\mathcal{A}(\pi)$	with
$\beta$	$(1, 0)$	$(1, 0, 0)$	$(1, 1, 1)$	$\{e_1 \pm e_2, 2e_1, 2e_2\}$	1	$\{(2, 2, 2)\}$	
					-1	$\{(2, 2, 1)\}$	$\beta \in \mathbb{C} \setminus \{0\}$
				$\{e_1 \pm e_2, 2e_1, -2e_2\}$	1	$\{(2, 0, 0)\}$	
					-1	$\{(2, 1, 0)\}$	$\beta \in \mathbb{C} \setminus \{0\}$
	$(0, -1)$	$(0, 0, -1)$	$(-1, -1, -1)$	$\{\pm e_1 - e_2, 2e_1, -2e_2\}$	1	$\{(0, 0, -2)\}$	
					-1	$\{(0, -1, -2)\}$	$\beta \in \mathbb{C} \setminus \{0\}$
				$\{\pm e_1 - e_2, -2e_1, -2e_2\}$	1	$\{(-2, -2, -2)\}$	
					-1	$\{(-1, -2, -2)\}$	$\beta \in \mathbb{C} \setminus \{0\}$

(2)  $\kappa = (0)$ ,  $\varepsilon = (1)$  by (F-2), and  $\lambda_d \sim (\beta, 1)$  with  $0 \leq \beta \in \mathbb{Z}$ . When  $\beta = 0$  this case coincides with the case (1), so we assume  $\beta \geq 1$ .

$\kappa_1$	$\varepsilon_1$	$\beta \in \mathbb{Z}$	$\lambda_d$	$\lambda_a$	$\frac{\rho(\mathbf{u} \cap \mathfrak{p})}{-\rho(\mathbf{u} \cap \mathfrak{k})}$	$\Psi$	$\mathcal{A}(\pi)$
0	1	1	$(1, -1)$	$(1, 0, -1)$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$\{e_1 \pm e_2, 2e_1, 2e_2\}$	$\{(2, 0, -1)\}$
						$\{\pm e_1 - e_2, 2e_1, 2e_2\}$	$\{(1, 0, -2)\}$
		$\geq 2$	$(\beta, 1)$	$(\beta, 1, 0)$	$(1, 2, 2)$	$\{e_1 \pm e_2, 2e_1, 2e_2\}$	$\{(\beta + 1, 3, 2)\}$
			$(-1, -\beta)$	$(0, -1, -\beta)$	$(-2, -2, -1)$	$\{\pm e_1 - e_2, -2e_1, -2e_2\}$	$\{(-2, -3, -\beta - 1)\}$
			$(\beta, -1)$	$(\beta, 0, -1)$	$(1, 0, 0)$	$\{e_1 \pm e_2, 2e_1, -2e_2\}$	$\{(\beta + 1, 0, -1)\}$
			$(1, -\beta)$	$(1, 0, -\beta)$	$(0, 0, -1)$	$\{\pm e_1 - e_2, 2e_1, -2e_2\}$	$\{(1, 0, -\beta - 1)\}$
		$\geq 2$	$(1, -1)$	$(1, 0, -1)$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$\{e_1 \pm e_2, 2e_1, 2e_2\}$	$\{(2, 0, -1)\}$
						$\{\pm e_1 - e_2, 2e_1, 2e_2\}$	$\{(1, 0, -2)\}$

(3)  $\kappa \sim (1)$ ,  $\lambda_d \sim (\beta, 0)$  with  $0 \leq \beta \in \mathbb{Z}$ . When  $\beta = 1$  this case coincides with (1), so we assume  $\beta \neq 1$ .

$\kappa_1$	$\beta \in \mathbb{Z}$	$\lambda_d$	$\lambda_a$	$\frac{\rho(\mathbf{u} \cap \mathbf{p})}{-\rho(\mathbf{u} \cap \mathbf{k})}$	$\Psi$	$\varepsilon_1$	$\mathcal{A}(\pi)$
1	0	(0, 0)	(0, 0, 0)	(0, 0, 0)	$\{e_1 \pm e_2, 2e_1, -2e_2\}$	1	$\{(1, 0, 0)\}$
						-1	$\{(1, 1, 0)\}$
					$\{\pm e_1 - e_2, 2e_1, -2e_2\}$	1	$\{(0, 0, -1)\}$
						-1	$\{(0, -1, -1)\}$
	$\geq 2$	$(\beta, 0)$	$(\beta, 0, 0)$	$(1, 1, 1)$	$\{e_1 \pm e_2, 2e_1, 2e_2\}$	1	$\{(\beta + 1, 2, 2)\}$
						-1	$\{(\beta + 1, 2, 1)\}$
					$\{e_1 \pm e_2, 2e_1, -2e_2\}$	1	$\{(\beta + 1, 0, 0)\}$
						-1	$\{(\beta + 1, 1, 0)\}$
		$(0, -\beta)$	$(0, 0, -\beta)$	$(-1, -1, -1)$	$\{\pm e_1 - e_2, 2e_1, -2e_2\}$	1	$\{(0, 0, -\beta - 1)\}$
						-1	$\{(0, -1, -\beta - 1)\}$
					$\{\pm e_1 - e_2, -2e_1, -2e_2\}$	1	$\{(-2, -2, -\beta - 1)\}$
						-1	$\{(-1, -2, -\beta - 1)\}$

C.3.  $(\mathbf{v}, \mathbf{s}, \mathbf{t}) = (\mathbf{1}, \mathbf{0}, \mathbf{2})$ .  $\varepsilon = (\varepsilon_1, \varepsilon_2)$ ,  $\kappa = (\kappa_1, \kappa_2)$ , and  $(\lambda_d \mid (\kappa_1, \kappa_2)) \sim (\beta, 0, 1)$ .

(F-2):  $\kappa_i = 0 \Rightarrow \varepsilon_i = -1$ ;  $\kappa_1 = \pm \kappa_2 \Rightarrow \varepsilon_1 = \varepsilon_2$ .

(1)  $\lambda_d = (0)$ . Up to permutations and sign-changes  $(\kappa_1, \kappa_2) = (\beta, 1)$ .

$(\kappa_1, \kappa_2)$	$\lambda_d$	$\frac{\lambda_a + \rho(\mathbf{u} \cap \mathbf{p})}{-\rho(\mathbf{u} \cap \mathbf{k})}$	$\Psi$	$(\varepsilon_1, \varepsilon_2)$	$\mathcal{A}(\pi)$	with
$(\beta, 1)$	(0)	(0, 0, 0)	$\{2e_1\}$	(1, 1)	$\{(1, 0, 0)\}$	$\beta \in \mathbb{C} \setminus \{\pm 1\}$
				(-1, -1)	$\{(1, 1, 1)\}$	
				(1, -1) or (-1, 1)	$\{(1, 1, 0)\}$	
			$\{-2e_1\}$	(1, 1)	$\{(0, 0, -1)\}$	$\beta \in \mathbb{C} \setminus \{\pm 1\}$
				(-1, -1)	$\{(-1, -1, -1)\}$	
				(1, -1) or (-1, 1)	$\{(0, -1, -1)\}$	

(2)  $\lambda_d \sim (1)$ . Up to permutations and sign-changes  $(\kappa_1, \kappa_2) = (\beta, 0)$ . By (F-2)  $\varepsilon_2 = -1$ .

$(\kappa_1, \kappa_2)$	$\varepsilon_2$	$\lambda_d$	$\Psi$	$\lambda_a$	$\frac{\rho(\mathbf{u} \cap \mathbf{p})}{-\rho(\mathbf{u} \cap \mathbf{k})}$	$\varepsilon_1$	$\mathcal{A}(\pi)$	with
$(\beta, 0)$	-1	(1)	$\{2e_1\}$	(1, 0, 0)	(1, 1, 1)	1	$\{(2, 2, 1), (2, 1, 0)\}$	$\beta \in \mathbb{C} \setminus \{0\}$
						-1	$\{(2, 1, 1)\}$	
		(-1)	$\{-2e_1\}$	(0, 0, -1)	(-1, -1, -1)	1	$\{(0, -1, -2), (-1, -2, -2)\}$	$\beta \in \mathbb{C} \setminus \{0\}$
						-1	$\{(-1, -1, -2)\}$	

(3)  $\lambda_d \sim (\beta)$  with  $0 \leq \beta \in \mathbb{Z}$ . Up to permutations and sign-changes  $(\kappa_1, \kappa_2) = (1, 0)$ . By (F-2)  $\varepsilon_2 = -1$ . When  $\beta = 0$  or 1, this case coincides with the cases (1) or (2), so we assume  $\beta \geq 2$ .

$(\kappa_1, \kappa_2)$	$\varepsilon_2$	$\beta \in \mathbb{Z}$	$\lambda_d$	$\Psi$	$\lambda_a$	$\frac{\rho(\mathbf{u} \cap \mathbf{p})}{-\rho(\mathbf{u} \cap \mathbf{k})}$	$\varepsilon_1$	$\mathcal{A}(\pi)$
(1, 0)	-1	$\geq 2$	$(\beta)$	$\{2e_1\}$	$(\beta, 0, 0)$	(1, 1, 1)	1	$\{(\beta + 1, 2, 1), (\beta + 1, 1, 0)\}$
							-1	$\{(\beta + 1, 1, 1)\}$
			$(-\beta)$	$\{-2e_1\}$	$(0, 0, -\beta)$	(-1, -1, -1)	1	$\{(0, -1, -\beta - 1), (-1, -2, -\beta - 1)\}$
							-1	$\{(-1, -1, -\beta - 1)\}$

C.4.  $(\mathbf{v}, \mathbf{s}, \mathbf{t}) = (\mathbf{0}, \mathbf{0}, \mathbf{3})$ .  $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ , and  $\kappa = (\kappa_1, \kappa_2, \kappa_3) \sim (\beta, 0, 1)$ .

(F-2):  $\kappa_i = 0 \Rightarrow \varepsilon_i = 1$ ;  $\kappa_i = \pm \kappa_j \Rightarrow \varepsilon_i = \varepsilon_j$ .

Up to permutations and sign-changes  $(\kappa_1, \kappa_2, \kappa_3) = (\beta, 1, 0)$ . By (F-2)  $\varepsilon_3 = 1$ .

$(\kappa_1, \kappa_2, \kappa_3)$	$\varepsilon_3$	$\lambda_a + \rho(\mathfrak{u} \cap \mathfrak{p})$ $-\rho(\mathfrak{u} \cap \mathfrak{k})$	$(\varepsilon_1, \varepsilon_2)$	$\mathcal{A}(\pi)$	with
$(\beta, 1, 0)$	1	$(0, 0, 0)$	$(1, 1)$	$\{(0, 0, 0)\}$	
			$(-1, -1)$	$\{(1, 1, 0), (0, -1, -1)\}$	$\beta \in \mathbb{C} \setminus \{0\}$
			$(1, -1)$	$\{(1, 0, 0), (0, 0, -1)\}$	$\beta \in \mathbb{C} \setminus \{\pm 1\}$
			$(-1, 1)$		$\beta \in \mathbb{C} \setminus \{0, \pm 1\}$

C.5.  $(\mathbf{v}, \mathbf{s}, \mathbf{t}) = (\mathbf{0}, \mathbf{1}, \mathbf{1})$ .  $\mu = (\mu_1)$ ,  $\nu = (\nu_1)$ ,  $\varepsilon = (\varepsilon_1)$ ,  $\kappa = (\kappa_1)$ , and  $(\frac{\mu_1 + \nu_1}{2}, \frac{-\mu_1 + \nu_1}{2}, \kappa_1) \sim (\beta, 0, 1)$ .

(F-2):  $\nu_1 = 0 \Rightarrow \mu_1$  is odd;  $\kappa_1 = 0 \Rightarrow \varepsilon_1 = 1$ .

(1)  $\kappa_1 = \beta$ , and  $(\frac{\mu_1 + \nu_1}{2}, \frac{-\mu_1 + \nu_1}{2}) \sim (0, 1)$ . Then  $\mu_1 = 1$ ,  $\nu_1 = 1$ .

$\mu_1$	$\nu_1$	$\lambda_a$	$\rho(\mathfrak{u} \cap \mathfrak{p})$ $-\rho(\mathfrak{u} \cap \mathfrak{k})$	$\kappa_1$	$\varepsilon_1$	$\mathcal{A}(\pi)$	with
1	1	$(\frac{1}{2}, 0, -\frac{1}{2})$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$\beta$	1	$\{(1, 0, -1)\}$	$0 \leq \beta \in \mathbb{Z}$
					-1	$\{(1, 1, -1), (1, -1, -1)\}$	$1 \leq \beta \in \mathbb{Z}$

(2)  $\kappa_1 = 1$ , and  $(\frac{\mu_1 + \nu_1}{2}, \frac{-\mu_1 + \nu_1}{2}) \sim (\beta, 0)$ . Then  $0 \leq \mu_1 = \beta = \nu_1$  with  $0 \leq \beta \in \mathbb{Z}$ , and  $\beta \neq 0$  by (F-2). When  $\beta = 1$  this case coincides with the case (1), so we assume  $\beta \geq 2$ .

$\kappa_1$	$\mu_1 = \nu_1$ $= \beta$	$\lambda_a$	$\rho(\mathfrak{u} \cap \mathfrak{p})$ $-\rho(\mathfrak{u} \cap \mathfrak{k})$	$\varepsilon_1$	$\mathcal{A}(\pi)$	with
1	$2m + 1$	$(m + \frac{1}{2}, 0, -m - \frac{1}{2})$	$(\frac{1}{2}, 0, -\frac{1}{2})$	1	$\{(m + 1, 0, -m - 1)\}$	$1 \leq m \in \mathbb{Z}$
				-1	$\{(m + 1, 1, -m - 1), (m + 1, -1, -m - 1)\}$	
	$2m$	$(m, 0, -m)$	$(\frac{1}{2}, 0, -\frac{1}{2})$	1	$\{(m + 1, 0, -m), (m, 0, -m - 1)\}$	
				-1	$\{(m + 1, 1, -m), (m + 1, -1, -m), (m, 1, -m - 1), (m, -1, -m - 1)\}$	

(3)  $\kappa_1 = 0$ , and  $(\frac{\mu_1 + \nu_1}{2}, \frac{-\mu_1 + \nu_1}{2}) \sim (\beta, 1)$ . By (F-2)  $\varepsilon_1 = 1$ . As  $\mu_1 = |\beta \pm 1| \in \mathbb{Z}$ ,  $0 \leq \beta \in \mathbb{Z}$ . When  $\beta = 0$  this case coincides with the case (1), so we assume  $\beta \geq 1$ . Notice that in this case  $\mu_1$  and  $\nu_1$  have the same parity, so  $\nu_1 \neq 0$  by (F-2).

$\kappa_1$	$\varepsilon_1$	$\mu_1$	$\nu_1$	$\lambda_a$	$\beta$	$\rho(\mathfrak{u} \cap \mathfrak{p})$ $-\rho(\mathfrak{u} \cap \mathfrak{k})$	$\mathcal{A}(\pi)$	with
0	1	$\beta - 1$	$\beta + 1$	$(\frac{\beta-1}{2}, 0, -\frac{\beta-1}{2})$	1	$(0, 0, 0)$	$\{(1, 0, 0), (0, 0, -1)\}$	$1 \leq m \in \mathbb{Z}$
					$2m$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$\{(m, 0, -m)\}$	
					$2m + 1$		$\{(m + 1, 0, -m), (m, 0, -m - 1)\}$	
					$2m$		$\{(m + 1, 0, -m - 1)\}$	
		$\beta + 1$	$\beta - 1$	$(\frac{\beta+1}{2}, 0, -\frac{\beta+1}{2})$	$2m + 1$		$\{(m + 2, 0, -m - 1), (m + 1, 0, -m - 2)\}$	
					$2m$			

C.6.  $(\mathbf{v}, \mathbf{s}, \mathbf{t}) = (\mathbf{1}, \mathbf{1}, \mathbf{0})$ .  $\mu = (\mu_1)$ ,  $\nu = (\nu_1)$ , and  $(\lambda_d \mid (\frac{\mu_1 + \nu_1}{2}, \frac{-\mu_1 + \nu_1}{2})) \sim (\beta, 0, 1)$ .

(F-2):  $\nu_1 = 0 \Rightarrow \mu_1$  is odd.

(1)  $\lambda_d = (0)$ , and  $(\frac{\mu_1 + \nu_1}{2}, \frac{-\mu_1 + \nu_1}{2}) \sim (\beta, 1)$ . Then  $\mu_1 = |\beta \pm 1| \in \mathbb{Z}$ , and  $0 \leq \beta \in \mathbb{Z}$ . In this case  $\mu_1$  and  $\nu_1$  are integers with the same parity, so  $\nu_1 \neq 0$  by (F-2).

$\lambda_d$	$\beta$	$\mu_1$	$\nu_1$	$\lambda_a$	$\frac{\rho(\mathfrak{u} \cap \mathfrak{p})}{-\rho(\mathfrak{u} \cap \mathfrak{k})}$	$\Psi$	$\mathcal{A}(\pi)$	with
(0)	0	1	1	$(\frac{1}{2}, 0, -\frac{1}{2})$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$\{2e_1\}$	$\{(1, 1, -1)\}$	$1 \leq m \in \mathbb{Z}$
						$\{-2e_1\}$	$\{(1, -1, -1)\}$	
	1	0	2	$(0, 0, 0)$	$(0, 0, 0)$	$\{2e_1\}$	$\{(1, 1, 0)\}$	
						$\{-2e_1\}$	$\{(0, -1, -1)\}$	
	$2m$	$\beta - 1$	$\beta + 1$	$(\frac{\beta - 1}{2}, 0, -\frac{\beta - 1}{2})$	$(\frac{1}{2}, 0, -\frac{1}{2})$	$\{2e_1\}$	$\{(m, 1, -m)\}$	
						$\{-2e_1\}$	$\{(m, -1, -m)\}$	
	$2m + 1$					$\{2e_1\}$	$\{(m + 1, 1, -m), (m, 1, -m - 1)\}$	
						$\{-2e_1\}$	$\{(m + 1, -1, -m), (m, -1, -m - 1)\}$	
	$2m$	$\beta + 1$	$\beta - 1$	$(\frac{\beta + 1}{2}, 0, -\frac{\beta + 1}{2})$		$\{2e_1\}$	$\{(m + 1, 1, -m - 1)\}$	
						$\{-2e_1\}$	$\{(m + 1, -1, -m - 1)\}$	
	$2m + 1$					$\{2e_1\}$	$\{(m + 2, 1, -m - 1), (m + 1, 1, -m - 2)\}$	
						$\{-2e_1\}$	$\{(m + 2, -1, -m - 1), (m + 1, -1, -m - 2)\}$	

(2)  $\lambda_d \sim (1)$ , and  $(\frac{\mu_1 + \nu_1}{2}, \frac{-\mu_1 + \nu_1}{2}) \sim (\beta, 0)$ . Then  $0 \leq \beta = \mu_1 = \nu_1 \in \mathbb{Z}$ , and  $\beta \neq 0$  by (F-2).

$\mu_1 = \nu_1 = \beta$	$\lambda_d$	$\Psi$	$\lambda_a$	$\rho(\mathfrak{u} \cap \mathfrak{p})$ $-\rho(\mathfrak{u} \cap \mathfrak{k})$	$\mathcal{A}(\pi)$	with
1	(1)	$\{2e_1\}$	$(1, \frac{1}{2}, -\frac{1}{2})$	$(1, \frac{3}{2}, \frac{1}{2})$	$\{(2, 2, 0)\}$	
	(-1)	$\{-2e_1\}$	$(\frac{1}{2}, -\frac{1}{2}, -1)$	$(-\frac{1}{2}, -\frac{3}{2}, -1)$	$\{(0, -2, -2)\}$	
2	(1)	$\{2e_1\}$	$(1, 1, -1)$	$(1, 1, 0)$	$\{(2, 2, -1)\}$	
	(-1)	$\{-2e_1\}$	$(1, -1, -1)$	$(0, -1, -1)$	$\{(1, -2, -2)\}$	
$2m + 1$	(1)	$\{2e_1\}$	$(m + \frac{1}{2}, 1, -m - \frac{1}{2})$	$(\frac{1}{2}, 1, -\frac{1}{2})$	$\{(m + 1, 2, -m - 1)\}$	$1 \leq m \in \mathbb{Z}$
	(-1)	$\{-2e_1\}$	$(m + \frac{1}{2}, -1, -m - \frac{1}{2})$	$(\frac{1}{2}, -1, -\frac{1}{2})$	$\{(m + 1, -2, -m - 1)\}$	
$2m$	(1)	$\{2e_1\}$	$(m, 1, -m)$	$(\frac{1}{2}, 1, -\frac{1}{2})$	$\{(m + 1, 2, -m), (m, 2, -m - 1)\}$	$2 \leq m \in \mathbb{Z}$
	(-1)	$\{-2e_1\}$	$(m, -1, -m)$	$(\frac{1}{2}, -1, -\frac{1}{2})$	$\{(m + 1, -2, -m), (m, -2, -m - 1)\}$	

(3)  $\lambda_d \sim (\beta)$  with  $0 \leq \beta \in \mathbb{Z}$ , and  $(\frac{\mu_1 + \nu_1}{2}, \frac{-\mu_1 + \nu_1}{2}) \sim (0, 1)$ . So  $\mu_1 = \nu_1 = 1$ . When  $\beta = 0$  or 1, this case coincides with the cases (1) or (2), so we assume  $\beta \geq 2$ .

$\lambda_d$	$\Psi$	$\mu_1$	$\nu_1$	$\lambda_a$	$\rho(\mathfrak{u} \cap \mathfrak{p})$ $-\rho(\mathfrak{u} \cap \mathfrak{k})$	$\mathcal{A}(\pi)$	with
$(\beta)$	$\{2e_1\}$	1	1	$(\beta, \frac{1}{2}, -\frac{1}{2})$	$(1, \frac{3}{2}, \frac{1}{2})$	$\{(\beta + 1, 2, 0)\}$	$2 \leq \beta \in \mathbb{Z}$
$(-\beta)$	$\{-2e_1\}$			$(\frac{1}{2}, -\frac{1}{2}, -\beta)$	$(-\frac{1}{2}, -\frac{3}{2}, -1)$	$\{(0, -2, -\beta - 1)\}$	

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